ADDITIONAL EXERCISES

To Accompany

Johnsonbaugh/Pfaffenberger: Foundations of Mathematical Analysis

1. [Section 12] Let $\{a_n\}$ be a sequence of nonzero real numbers that converges to a nonzero limit. Prove that $\{a_{n+1}/a_n\}$ converges.

2. [Section 24] Give an example of a positive continuous function f on $[1,\infty)$ such that $\sum_{n=1}^{\infty} f(n)$ diverges, but $\sum_{n=1}^{\infty} a^n f(a^n)$ converges for all $a > 1$.

3. [Section 24] (Suggested by Paul Erdős. As far as we know, this problem is unsolved.) True or false? If f is a positive continuous function on $[1,\infty)$ such that $\sum_{n=1}^{\infty} f(n+\varepsilon)$ diverges for all $\varepsilon > 0$, then $\sum_{n=1}^{\infty} a^n f(a^n)$ diverges for some $a > 1$.

4. [Section 24] (Suggested by Peter Ungar. As far as we know, this problem is also unsolved.) True or false? If f is a positive continuous function on $[1,\infty)$ such that $\sum_{n=1}^{\infty} f(n\varepsilon)$ diverges for all $\varepsilon > 0$, then $\sum_{n=1}^{\infty} a^n f(a^n)$ diverges for some $a > 1$.

5. [Section 26] Add to Exercise 26.1:

   (k) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{(-1)^n + n}$

6. [Section 26] Find an absolutely convergent series $\sum_{n=1}^{\infty} a_n$ such that

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a_n^2.$$

7. [Section 26] Prove *Tannery's Theorem*: Suppose that

   (a) $\displaystyle\sum_{k=1}^{\infty} s_{k,n}$ converges for all $n \geq 1$.

   (b) $\displaystyle\lim_{n\to\infty} s_{k,n} = s_k$ for all $k \geq 1$.

   (c) $|s_{k,n}| \leq M_k$ for all $k \geq 1$, $n \geq 1$.

   (d) $\displaystyle\sum_{k=1}^{\infty} M_k$ converges.

   Then $\sum_{k=1}^{\infty} s_k$ converges and

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} s_{k,n} = \sum_{k=1}^{\infty} s_k.$$

8. [Section 28] Prove that $\sum_{i=1}^{\infty} a_n/n$ converges if and only if $\sum_{i=1}^{\infty} a_n/(n+1)$ converges. (It is *not* assumed that $a_n \geq 0$ for all n.)

9. [Section 29] Prove that if $\lim_{n\to\infty} a_n = 0$, there exists a sequence $\{b_n\}$, where each $b_n = \pm 1$, such that $\sum_{n=1}^{\infty} a_n b_n$ converges.

10. [Section 29] Find the sum by rows and the sum by columns of the double series $\sum_{m,n} a_{m,n}$, where
$$a_{m,n} = \frac{1}{n^m}, \quad m, n \geq 2.$$

11. [Section 29] (This result is due to Uri Elias.) Let $\sum_{n=1}^{\infty} a_n$ be a convergent series, which is not necessarily *absolutely* convergent, and s be a positive integer. Suppose that $\sum_{n=1}^{\infty} a_{f(n)}$ is a rearrangement of $\sum_{n=1}^{\infty} a_n$ in which $n - f(n) \leq s$ for all n. (In words, each term in the original series that is shifted forward in the rearranged series is shifted at most s positions.) Prove that $\sum_{n=1}^{\infty} a_{f(n)}$ converges and
$$\sum_{n=1}^{\infty} a_{f(n)} = \sum_{n=1}^{\infty} a_n.$$

12. [Section 38] Let X be a subset of $\mathbf{R}$ with a countable number of limit points. Prove that X is countable.

13. [Section 42] Prove that if $\mathcal{U}$ is an open cover of a compact metric space (M, d), there exists $\delta > 0$ such that for each $x \in M$, there exists $U \in \mathcal{U}$ satisfying
$$\{y \mid d(x, y) < \delta\} \subseteq U.$$
The number δ is called a *Lebesgue number* for $\mathcal{U}$.

14. [Section 49] Let $\{a_n\}$ and $\{b_n\}$ be sequences such that $b_{n+1} = a_n + a_{n+1}$ for every positive integer n, and suppose that $\{b_n\}$ converges. Let ε satisfy $0 < \varepsilon < 1$. Give an example to show that $\{a_n/n^{\varepsilon}\}$ need not converge. (Compare with Exercise 20.22.)

15. [Section 54] Give an example of a nonnegative function f on $[0, 1]$ such that $f \in BV[0, 1]$, but $\sqrt{f} \notin BV[0, 1]$.

16. [Section 60] Suppose that $\{f_n\}$ is a sequence of continuous functions that converges pointwise to f on $\mathbf{R}$.
    (a) Prove that the set of points at which f is continous is dense in $\mathbf{R}$.
    (b) Prove that the set of points at which f is continous is a G_{δ} set.

17. [Section 61] Prove that if $\{f_n\}$ and f are continuous on $[a, b]$ and differentiable on (a, b), $\{f_n'\}$ converges uniformly to f' on (a, b), and $\lim_{n\to\infty} f_n(a) = f(a)$, then $\{f_n\}$ converges uniformly to f on $[a, b]$.

18. [Section 61] Let $\{f_n\}$ be a sequence of functions, continuous on $[a, b]$ and differentiable on (a, b). Prove that if $\{f_n'\}$ converges uniformly to some function g on (a, b) and $\lim_{n\to\infty} f_n(c)$ exists for some $c \in (a, b)$, then $\{f_n\}$ converges uniformly to some function f on $[a, b]$ and $f'(x) = g(x)$ for all $x \in (a, b)$.

19. [Section 66] (Contributed by Ken Ross.) Prove that $\exp(x)\exp(y) = \exp(x + y)$, for all x and y, assuming only that $\exp(0) = 1$ and $\exp'(x) = \exp(x)$, for all x. *Hint*: Differentiate $\exp(x)\exp(-x)$ to prove that $\exp(x)\exp(-x) = 1$, for all x. Conclude that $\exp(x) \neq 0$, for all x. Now show that for any a, $\exp(x + a)/\exp(x) = \exp(a)$, for all x, by differentiating $\exp(x + a)/\exp(x)$.

20. [Section 66] Use Tannery's Theorem (see Exercise 7) to prove that

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = \sum_{k=0}^{\infty}\frac{x^k}{k!}.$$

21. [Section 77] Show that $\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}$.