This help document accompanies Richard Johnsonbaugh: *Discrete Mathematics*, 6th edition, Prentice Hall, Upper Saddle River, N.J., 2005.

WebHelp: Mathematical Induction

Mathematical induction is used to prove a sequence of statements indexed by the positive integers. For example, if

$$S(n) : 1 + 2 + \cdots + n = \frac{n(n+1)}{2},$$

mathematical induction can be used to prove that $S(n)$ is true for all positive integers n. In other words, mathematical induction can be used to prove that

$$\begin{aligned}
S(1): &\quad 1 = \frac{1 \cdot 2}{2}, \\
S(2): &\quad 1 + 2 = \frac{2 \cdot 3}{2}, \\
S(3): &\quad 1 + 2 + 3 = \frac{3 \cdot 4}{2}, \\
&\vdots \\
S(n): &\quad 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}, \\
S(n+1): &\quad 1 + 2 + 3 + \cdots + n + (n+1) = \frac{(n+1)(n+2)}{2}, \\
&\vdots
\end{aligned}$$

are all true. [Notice that $S(n+1)$ is obtained from $S(n)$ by everywhere substituting $n+1$ for n.]

In its simplest form (we discuss the *strong* form of mathematical induction at the end of this WebHelp), mathematical induction requires two steps

- **Basis Step.** Prove that $S(1)$ is true.

- **Inductive Step.** For every n, *assume* that $S(n)$ is true and *prove* that $S(n+1)$ is true.

To see why the Basis and Inductive Steps prove that $S(n)$ is true for all n, consider any specific value of n, for example, $n = 5$. Is $S(5)$ true? Well, because of the Basis Step, $S(1)$ is true. The Inductive Step says that if $S(1)$ is true, then $S(2)$ is true. $S(1)$ is true! Therefore $S(2)$ is true. The Inductive Step says that if $S(2)$ is true, then $S(3)$ is true. $S(2)$ is true! Therefore $S(3)$ is true. The Inductive Step says that if $S(3)$ is true, then $S(4)$ is true. $S(3)$ is true! Therefore $S(4)$ is true. The Inductive Step says that if $S(4)$ is true, then $S(5)$ is true. $S(4)$ is true! Therefore $S(5)$ is true! We could use a similar argument *for any value of n*, therefore $S(n)$ *is true for every n*.

The Basis Step is usually straightforward. In the previous example, the Basis Step is to prove

$$1 = \frac{1 \cdot 2}{2},$$

which is certainly true! In the previous example, the Inductive Step is to *assume* that

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

is true, and then prove that

$$1 + 2 + \cdots + n + (n+1) = \frac{(n+1)(n+2)}{2}$$

is true. *We recommend that you always write out both case n and case $n+1$ before proceeding further with the Inductive Step.*

The key to proving the Inductive Step is to "uncover" case n within case $n+1$. Although the meaning of "uncover" depends on the context, it is *always* the case that the success of the Inductive Step rests on uncovering case n within case $n+1$.

For our example, case n involves

$$1 + 2 + \cdots + n,$$

which appears within case $n+1$:

$$\underbrace{1 + 2 + \cdots + n}_{\text{This is case } n.} + (n+1).$$

Since we are assuming that case n is true, that is, that

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2},$$

we may substitute

$$\frac{n(n+1)}{2}$$

for

$$1 + 2 + \cdots + n$$

in case $n+1$ to obtain

$$1 + 2 + \cdots + n + (n+1) = \frac{n(n+1)}{2} + (n+1).$$

[Notice that the term $(n+1)$ was not replaced by anything and, so, is just copied from the left side to the right side of the equation.] The Inductive Step is completed by using algebra to get the right side into the correct form, namely,

$$\frac{(n+1)(n+2)}{2}.$$

We have

$$\begin{aligned} 1 + 2 + \cdots + n + (n+1) &= \underbrace{\frac{n(n+1)}{2} + (n+1)}_{\text{factor out } n+1} \\ &= (n+1)\left(\frac{n}{2} + 1\right) \\ &= \frac{(n+1)(n+2)}{2}. \end{aligned}$$

The Inductive Step is finished and the proof by mathematical induction is complete.

Summary

To give a proof that $S(n)$ is true for every positive integer n using mathematical induction in its simplest form:

- Prove directly that $S(1)$ is true.
- *Assume* that $S(n)$ is true and *prove* that $S(n+1)$ is true. As an aid, write out $S(n)$ and $S(n+1)$ explicitly, remembering that $S(n+1)$ is obtained from $S(n)$ by everywhere replacing n by $n+1$.

To help with the latter step, look for $S(n)$ within $S(n+1)$.

Strong Form of Mathematical Induction

In the strong form of mathematical induction, the preceding Inductive Step

- **Inductive Step.** For every n, *assume* that $S(n)$ is true and *prove* that $S(n+1)$ is true.

is replaced by

- **Inductive Step for Strong Form of Mathematical Induction.** For every n, *assume* that $S(k)$ is true *for all* $k < n$ and *prove* that $S(n)$ is true.

The Basis Step is unchanged. In the strong form of mathematical induction, to prove that $S(n)$ is true we may assume the truth of $S(k)$ *for all* k *that precede* n, namely $1, 2, \ldots, n-1$. In the simple form of mathematical induction, to prove that $S(n+1)$ is true, we assume the truth of $S(k)$ *only for the* k *that immediately precedes* $n+1$, namely n. In other words, in the strong form of mathematical induction, to prove that $S(n)$ is true, we assume that $S(1), S(2), \ldots, S(n-1)$ are all true. In the simple form of mathematical induction, to prove that $S(n+1)$ is true, we assume only that $S(n)$ is true. The strong form of mathematical induction is useful when the truth of *all* the preceding cases helps prove case n.