This help document accompanies Richard Johnsonbaugh: *Discrete Mathematics*, 6th edition, Prentice Hall, Upper Saddle River, N.J., 2005.

WebHelp: Growth of Functions

The theta notation characterizes exactly the rate of growth of a function ignoring constant coefficients and a finite number of exceptions. Intuitively, in an expression such as

$$t(n) = n^2 + n \lg n + 2,$$

the theta notation for $t(n)$ is the dominant term; after all, the dominant term is the term that dominates, and thus determines, the rate of growth. In this case, the dominant term is n^2 and so $t(n) = \Theta(n^2)$.

By definition, $f(n) = \Theta(g(n))$ if there exist positive constants C_1, C_2, and N such that

$$C_1|g(n)| \leq |f(n)| \leq C_2|g(n)|$$

for all $n \geq N$. (The inequality is allowed to fail for a finite number of n preceding N.) If f and g are positive (e.g., f and g are measures of time), the absolute value bars can be ignored.

Example. Let's *prove* that if

$$t(n) = n^2 + n \lg n + 2,$$

then $t(n) = \Theta(n^2)$.

Let's first find a constant C_2 satisfying

$$n^2 + n \lg n + 2 \leq C_2 n^2.$$

To do so, we must bound $n^2 + n \lg n + 2$ from above. Here the idea is to replace $n^2 + n \lg n + 2$ by something *larger*. We do so by replacing each term in the expression by a larger or equal term—namely n^2. It's in this sense that n^2 is the dominant term in the expression $n^2 + n \lg n + 2$.

Since

$$\lg n \le n$$

for all $n \ge 1$,

$$n \lg n \le n^2$$

for all $n \ge 1$. For all $n \ge 2$,

$$2 \le n^2;$$

therefore

$$n^2 + n \lg n + 2 \le n^2 + n^2 + n^2 = 3n^2$$

for all $n \ge 2$. Thus if we take $C_2 = 3$, we have

$$n^2 + n \lg n + 2 \le C_2 n^2.$$

for all $n \ge 2$.

We must also find a constant C_1 satisfying

$$C_1 n^2 \le n^2 + n \lg n + 2.$$

This time we must bound $n^2 + n \lg n + 2$ from below. Here the idea is to replace $n^2 + n \lg n + 2$ by something *smaller*. We do so by replacing each term in the expression by a smaller or equal term. Notice that since $n \lg n$ and 2 are nonnegative for all $n \ge 1$, we may replace them by zero leaving only n^2; that is,

$$n^2 \le n^2 + n \lg n + 2$$

for all $n \ge 1$. Thus if we take $C_1 = 1$,

$$C_1 n^2 \le n^2 + n \lg n + 2$$

for all $n \ge 1$. Therefore

$$C_1 n^2 \le t(n) \le C_2 n^2$$

for all $n \ge 2$. It follows from the definition that $t(n) = \Theta(n^2)$. □

Among the various estimates for a function, theta notation is the most desirable since it gives both an upper *and* a lower bound. Big oh notation provides only an upper bound, and omega notation provides only a lower bound.

Example. There are lots of big oh notations for

$$t(n) = n^2 + n \lg n + 2.$$

In the previous example, we showed that

$$t(n) \leq 3n^2$$

for all $n \geq 2$. Thus $t(n) = O(n^2)$. Since $n^2 \leq n^3$ for all $n \geq 2$, it follows that

$$t(n) \leq 3n^3$$

for all $n \geq 2$. Thus $t(n) = O(n^3)$. Similarly, $t(n)$ is big oh of any function "larger" than n^2. We have

$$t(n) = O(n^4), \qquad t(n) = O(2^n), \qquad t(n) = O(n!).$$

□

Example. There are lots of omega notations for

$$t(n) = n^2 + n \lg n + 2.$$

In the previous example, we showed that

$$t(n) \geq n^2$$

for all $n \geq 1$. Thus $t(n) = \Omega(n^2)$. Since $n^2 \geq n \lg n$ for all $n \geq 1$, it follows that

$$t(n) \geq n \lg n$$

for all $n \geq 1$. Thus $t(n) = \Omega(n \lg n)$. Similarly, $t(n)$ is omega of any function "smaller" than n^2. We have

$$t(n) = \Omega(n), \qquad t(n) = \Omega(\sqrt{n}), \qquad t(n) = \Omega(1).$$

□