This help document accompanies Richard Johnsonbaugh: *Discrete Mathematics*, 5th edition, Prentice Hall, Upper Saddle River, N.J., 2001.

WebHelp: Matrices of Relations

If R is a relation from X to Y and $x_1, \ldots, x_m$ is an ordering of the elements of X and $y_1, \ldots, y_n$ is an ordering of the elements of Y, the matrix A of R is obtained by defining $A_{ij} = 1$ if $x_i R y_j$ and 0 otherwise. Note that the matrix of R depends on the orderings of X and Y.

Example. The matrix of the relation

$$R = \{(1, a), (3, c), (5, d), (1, b)\}$$

from $X = \{1, 2, 3, 4, 5\}$ to $Y = \{a, b, c, d, e\}$ relative to the orderings $1, 2, 3, 4, 5$ and a, b, c, d, e is

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

□

What the Matrix of a Relation Tells Us

Let R be a relation, and let A be its matrix relative to some orderings. By definition, an element (x_i, y_j) is in R if and only if $A_{ij} = 1$. The domain of R consists of all elements x_i for which row i in A contains at least one 1. The range of R consists of all elements x_j for which column j in A contains at least one 1.

Example. We see from the matrix in the first example that the elements $(1, a), (3, c), (5, d), (1, b)$ are in the relation because those entries in the matrix are 1. We also see that the domain is $\{1, 3, 5\}$ because those rows contain at least one 1, and the range is $\{a, b, c, d\}$ because those columns contain at least one 1. □

Let R be a relation on a set X, let $x_1, \ldots, x_n$ be an ordering of X, and let A be the matrix of R where the ordering $x_1, \ldots, x_n$ is used for both the rows and columns. Then R is reflexive if and only if the main diagonal of A consists of all 1's (i.e., $A_{ii} = 1$ for all i). R is symmetric if and only if A is symmetric (i.e., $A_{ij} = A_{ji}$ for all i and j). R is antisymmetric if and only if for all $i \neq j$, A_{ij} and A_{ji} are not both equal to 1. R is transitive if and only if whenever A^2_{ij} is nonzero, A_{ij} is also nonzero.

Example. The matrix of the relation

$$R = \{(1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 3), (4, 3)\}$$

on $\{1, 2, 3, 4\}$ relative to the ordering $1, 2, 3, 4$ is

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

We see that R is not reflexive because A's main diagonal contains a 0. R is not symmetric because A is not symmetric; for example, $A_{12} = 1$, but $A_{21} = 0$. R is antisymmetric because for all $i \neq j$, A_{ij} and A_{ji} are not both equal to 1. The square of A is

$$A^2 = \begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

R is transitive because whenever A^2_{ij} is nonzero, A_{ij} is also nonzero. □

Composition Corresponds to Multiplication

Let R be a relation from X to Y, and let S be a relation from Y to Z. Choose orderings for X, Y, and Z; all matrices are with respect to these orderings. Let A be the matrix of R, and let B be the matrix of S. Then the matrix of $S \circ R$ is obtained by changing each nonzero entry in the matrix product AB to 1.

Example. Consider the relations

$$R = \{(1, a), (1, b), (1, d), (2, a), (3, c), (4, b)\}$$

from $X = \{1, 2, 3, 4\}$ to $Y = \{a, b, c, d\}$, and

$$S = \{(a, x), (c, y), (d, x), (d, y)\}$$

from Y to $Z = \{x, y\}$. Using the orderings $1, 2, 3, 4$; a, b, c, d; and x, y; the matrix of R is

$$A = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

and the matrix of S is

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

The matrix product is

$$AB = \begin{pmatrix} 2 & 1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Setting all nonzero entries to 1 in the previous matrix gives

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix},$$

which is the matrix of the relation

$$S \circ R = \{(1, x), (1, y), (2, x), (3, y)\}.$$

$\square$