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# Triangular Dirichlet Kernels and Growth of $L^p$ Lebesgue Constants

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**Abstract** Let $P$ be a polygon in $\mathbb{Z}^2$ and consider the mapping of an $L^1(\mathbb{T}^2)$ function into the partial sum of its Fourier series determined by the dilate of $P$ by the integer $N$. If the image space is endowed with the $L^p$ norm, $1 < p < \infty$, then the operator norm will be given by the $L^p$ norm of $\sum_{(m,n)\in NP} e^{2\pi i(mx+ny)}$. The size of this operator norm is shown to be $O(N^{2(1-1/p)})$ when the polygon is a triangle. The estimate is independent of the shape of the triangle. For a $k$ sided polygon the corresponding estimate is $O(kN^{2(1-1/p)})$.

**Keywords** Lebesgue constant · Dirichlet kernels in two dimensions · Dirichlet kernels for polygons

**Mathematics Subject Classification (2000)** Primary 42B15 · 42A05 · Secondary 42A45

## 1 Introduction

In one dimension there is a very well known estimate [9]

$$\int_0^1 \left| \sum_{n=1}^{N} e^{2\pi n i x} \right| dx \backsimeq \frac{4}{\pi^2} \ln N. \tag{1}$$

This integral is called the Lebesgue constant. In higher dimensions there are as many Lebesgue constants as there are generalizations of an interval of integers,

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$\{1, 2, \ldots, N\}$. One such generalization is the set of two dimensional integer lattice points lying in the right triangle

$$K_{\theta,N} = \left\{x \in \mathbb{E}^2 : x_1 + \theta^{-1}x_2 \le N, x_1 \ge 0, x_2 \ge 0\right\},$$

where $\mathbb{E}$ is the real numbers and $\theta \in (0, 1]$. In [8], A.A. Yudin and V.A. Yudin produce an estimate similar to relation (1) for this case. Reference [6] is also relevant here. When $p \in (1, \infty)$, the $L^1$ relation (1) has a well known $L^p$ analogue, namely

$$\int_0^1 \left|\sum_{n=1}^{N} e^{2\pi nix}\right|^p dx \backsimeq \left(\frac{2}{\pi}\int_0^\infty \left|\frac{\sin u}{u}\right|^p du\right) N^{p-1}. \tag{2}$$

A sharp form of this appears as Lemma 1 of [1]. In Theorem 1 we extend what Yudin and Yudin did for the triangle $K_{\theta,N}$ and the $L^1$ norm to the case of the same triangle $K_{\theta,N}$ and the $L^p$ norm, for every finite $p > 1$. The extension is routine except at one point where the easily verified inequality

$$\left|\frac{d}{dx}\left(\sum_{n=1}^{N} e^{2\pi nix}\right)\right| \le C\frac{N}{|x|}, \quad 0 < |x| \le \frac{1}{2}$$

proves to be useful.

An immediate corollary for an estimate of an $L^1$ or an $L^p$ Lebesgue constant for a triangle $K_{\theta,N}$ is a similar estimate for any two dimensional polygon. In the $L^1$ case, results for all polyhedrons in all dimensions are already known. See Chap. 6 of [5] for this. In Corollary 2 we state the $L^p$ result for polygons which follows immediately from Theorem 1. Also results for all dimensions which include those found here when restricted to dimension two have been published recently [3]. However in contrast to the proof given here, the methods employed in [3] are not at all number theoretic. Also the bounds established in [3], when specialized to two dimensions, are less precise than the bounds found here.

My original interest in this question came from studying a fundamental question from functional analysis: Is a commuting with translation operator of weak restricted type (2, 2) necessarily bounded on $L^2(\mathbb{Z})$? The affirmative answer to that question had a strong connection to this question: Given a subset of [0, 1] of very small measure, can one find a sum of exponentials with coefficients 0 or 1 whose $L^p$ norm is largely concentrated on that subset? The result of this paper played an important role in the positive resolution of the second question. See [2] for explanations of these questions and their connections.

## 2 Results

For $n = (n_1, n_2) \in \mathbb{Z}^2$, let $e(nx)$ denote $\exp(2\pi i(n_1x_1 + n_2x_2))$.

**Theorem 1** *Let $\mathbb{E}$ be the real numbers, $\theta \in (0, 1]$ and*

$$K_{\theta,N} = \left\{x \in \mathbb{E}^2 : x_1 + \theta^{-1}x_2 \le N, x_1 \ge 0, x_2 \ge 0\right\}.$$

*Then for each finite $p > 1$ and arbitrary $N \geq 1$,*

$$\int_{\mathbb{T}^2} \left| \sum_{n \in K_{\theta,N} \cap \mathbb{Z}^2} e(nx) \right|^p dx \leq C_p N^{2p-2}$$

*uniformly with respect to $\theta$ and $N$.*

**Corollary 2** *Let $P$ be a $k$ sided polygon in $\mathbb{E}^2$ of diameter $N$. Then*

$$\int_{\mathbb{T}^2} \left| \sum_{n \in P \cap \mathbb{Z}^2} e(nx) \right|^p dx \leq k C_p N^{2p-2}.$$

The corollary is almost immediate since the polygon can be decomposed into $k-2$ triangles. See Theorem 2 in [8] for the details. So we pass immediately to the proof of the theorem.

*Proof* Upon setting $M = \lfloor N \rfloor$, we have

$$\Delta_{\theta,N}(x) = \sum_{n \in K_{\theta,N} \cap \mathbb{Z}^2} e(nx) = \sum_{n_1=0}^{M} e(n_1 x_1) \sum_{n_2=0}^{\lfloor \theta(N-n_1) \rfloor} e(n_2 x_2)$$

since the upper edge of the triangle defining $K$ is given by $x_1 + \theta^{-1} x_2 = N$, or $x_2 = \theta(N - x_1)$. The inner sum is a geometric series since $e(n_2 x_2) = e(x_2)^{n_2}$ so that we have

$$\sum_{n_2=0}^{\lfloor \theta(N-n_1) \rfloor} e(n_2 x_2) = \frac{e((\lfloor \theta(N-n_1) \rfloor + 1) x_2) - 1}{e(x_2) - 1}$$

and

$$\Delta_{\theta,N}(x) = \sum_{n_1=0}^{M} e(n_1 x_1) \frac{e((\lfloor \theta(N-n_1) \rfloor + 1) x_2) - 1}{e(x_2) - 1}.$$

Since

$$(\lfloor \theta(N-n_1) \rfloor + 1) = \theta(N-n_1) + 1 - \{\theta(N-n_1)\},$$

where here, and hereafter, $\{\cdots\}$ will be reserved to mean fractional part;

$$\Delta_{\theta,N}(x) = \sum_{n_1=0}^{M} e(n_1 x_1) \frac{e((\theta(N-n_1) + 1) x_2 - \{\theta(N-n_1)\} x_2) - 1}{e(x_2) - 1}.$$

But

$$\begin{aligned}
&e\left(\left(\theta\left(N-n_1\right)+1\right)x_2-\left\{\theta\left(N-n_1\right)\right\}x_2\right)-1\\
&\quad=e\left(\left(\theta\left(N-n_1\right)+1\right)x_2\right)-1\\
&\qquad+e\left(\left(\theta\left(N-n_1\right)+1\right)x_2-\left\{\theta\left(N-n_1\right)\right\}x_2\right)\\
&\qquad-e\left(\left(\theta\left(N-n_1\right)+1\right)x_2\right)\\
&\quad=e\left(\left(\theta\left(N-n_1\right)+1\right)x_2\right)-1\\
&\qquad=e\left(\left(\theta\left(N-n_1\right)+1\right)x_2\right)\left(e\left(-\left\{\theta\left(N-n_1\right)\right\}x_2\right)-1\right),
\end{aligned}$$

so

$$\begin{aligned}
&e\left(n_1x_1\right)\left(e\left(\left(\theta\left(N-n_1\right)+1\right)x_2-\left\{\theta\left(N-n_1\right)\right\}x_2\right)-1\right)\\
&\quad=e\left(n_1x_1\right)\left(e\left(\left(\theta\left(N-n_1\right)+1\right)x_2\right)-1\right)\\
&\qquad+e\left(n_1\left(x_1-\theta x_2\right)+(\theta N+1)x_2\right)\left(e\left(-\left\{\theta\left(N-n_1\right)\right\}x_2\right)-1\right),
\end{aligned}$$

and

$$\begin{aligned}
\Delta_{\theta,N}(x)&=\sum_{n_1=0}^{M}e\left(n_1x_1\right)\frac{e\left(\left(\theta\left(N-n_1\right)+1\right)x_2\right)-1}{e\left(x_2\right)-1}\\
&\qquad+\sum_{n_1=0}^{M}e\left(n_1\left(x_1-\theta x_2\right)+(\theta N+1)x_2\right)\frac{e\left(-\left\{\theta\left(N-n_1\right)\right\}x_2\right)-1}{e\left(x_2\right)-1}\\
&=J_1(x)+J_2(x).
\end{aligned}$$

We first estimate

$$I_1=\int_{\mathbb{T}^2}\left|J_1(x)\right|^p dx.$$

For the Dirichlet kernel $D_n(x)=\sum_{\nu=0}^{n}e(\nu x)=e(\frac{nx}{2})\frac{\sin(\pi(n+1)x)}{\sin\pi x}$, we use the estimates

D1 $|D_n(x)|\le n+1\ll n$,
D2 $|D_n(x)|\le\frac{1}{|2x|}\ll|x|^{-1}$ if $0<|x|\le 1/2$,
D3 $|D_n'(x)|\le\pi n(n+1)\ll n^2$,
D4 $|D_n'(x)|\ll n|x|^{-1}$ if $0<|x|\le 1/2$.

Estimate D2 uses $\sin\pi x\ge 2x$ on $[0,1/2]$. Estimate D3 follows from $|D_n'|=|\sum_{\nu=1}^{n}2\pi i\nu e(\nu x)|\le 2\pi\sum_{\nu=1}^{n}\nu$. For D4, differentiate the closed form of $D_n$ factor out $e(\frac{nx}{2})$ and then take $|e(\frac{nx}{2})|=1$ into account to get for $0<|x|\le 1/2$,

$$\begin{aligned}
\left|D_n'(x)\right|&=\left|\pi in\frac{\sin\pi\left(n+1\right)x}{\sin\pi x}-\pi\frac{\cos\pi x}{\sin^2\pi x}\sin\pi\left(n+1\right)x\right.\\
&\qquad\left.+\frac{\cos\pi\left(n+1\right)x}{\sin\pi x}\pi\left(n+1\right)\right|\\
&\le\left|\pi n\frac{1}{2x}\right|+\left|\pi\frac{1}{(2x)^2}\pi\left(n+1\right)x\right|+\left|\frac{1}{2x}\pi\left(n+1\right)\right|\ll\frac{n}{|x|}.
\end{aligned}$$

Returning to the estimate of $I_1$, from

$$\begin{aligned} &e(n_1x_1)(e((\theta(N-n_1)+1)x_2)-1) \\ &\quad = e((\theta N+1)x_2)\,e(n_1(x_1-\theta x_2)) - e(n_1x_1) \end{aligned}$$

and $1/|e(x_2)-1| \ll |x_2|^{-1}$ follows

$$I_1 \ll \int_{\mathbb{T}^2} |x_2|^{-p}\,|e((\theta N+1)x_2)\,D_M(x_1-\theta x_2) - D_M(x_1)|^p\,dx. \tag{3}$$

Since

$$\begin{aligned} &|e((\theta N+1)x_2)\,D_M(x_1-\theta x_2) - D_M(x_1)| \\ &\quad = |e((\theta N+1)x_2)\,D_M(x_1-\theta x_2) - D_M(x_1-\theta x_2) \\ &\qquad + D_M(x_1-\theta x_2) - D_M(x_1)\,| \\ &\quad \le |e((\theta N+1)x_2)-1|\,|D_M(x_1-\theta x_2)| + |D_M(x_1-\theta x_2) - D_M(x_1)| \\ &\quad = |2\sin\pi(\theta N+1)x_2|\cdot|D_M(x_1-\theta x_2)| + |D_M(x_1-\theta x_2) - D_M(x_1)|. \end{aligned} \tag{4}$$

So to bound $I_1$, it suffices to bound $I_A$ and $I_B$, where

$$\begin{aligned} I_A &= \int_{\mathbb{T}^2} |x_2|^{-p}\left|2\sin\frac{1}{2}(\theta N+1)x_2\right|^p |D_M(x_1-\theta x_2)|^p\,dx, \quad \text{and} \\ I_B &= \int_{\mathbb{T}^2} |x_2|^{-p}\,|D_M(x_1-\theta x_2) - D_M(x_1)|^p\,dx. \end{aligned}$$

First, the change of variable $x_1' = x_1 - \theta x_2$, $x_2' = x_2$ has Jacobian 1, so

$$I_A = \int_{\mathbb{T}} |D_M(x_1)|^p\,dx_1 \int_{\mathbb{T}} |x_2|^{-p}\,|2\sin\pi(\theta N+1)x_2|^p\,dx_2.$$

The first integral is $O(N^{p-1})$ by Lemma 2 of [1], and the substitution $u = \pi(\theta N+1)x_2$ shows that the second integral is

$$\begin{aligned} &\left(\int_0^\infty - \int_{\pi(\theta N+1)}^\infty\right)\left|\frac{u}{\pi(\theta N+1)}\right|^{-p} |2\sin u|^p\,\frac{du}{\pi(\theta N+1)} \\ &\quad = c_p(\theta N+1)^{p-1} - E_{p,\theta}(N) \end{aligned}$$

where $c_p = \int_0^\infty (\frac{2\pi\sin u}{u})^p \frac{du}{\pi}$ and $|E_{p,\theta}(N)| \ll (\theta N+1)^{p-1}\int_{\pi(\theta N+1)}^\infty u^{-p}du = \frac{\pi^{-p+1}}{p-1}$. So uniformly in $\theta$,

$$I_A = O_p\left(N^{2p-2}\right).$$

We now turn to the problem of estimating $I_B$. For estimating $I_B$ we change notation from $(x_1, x_2)$ to $(x, y)$ and from $dx$ to $dxdy$. We must estimate

$$I_B = \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} |y|^{-p}\,|D_M(x-\theta y) - D_M(x)|^p\,dxdy.$$

We make two reductions. First it is enough to integrate over $[0, 1/2]^2$. This is because symmetry considerations show that $\int_0^{1/2}\int_0^{1/2} = \int_{-1/2}^{0}\int_{-1/2}^{0}$ and that the integrals over the other two quadrants are both equal to

$$\int_0^{1/2}\int_0^{1/2} |y|^{-p} |D_M(x+\theta y) - D_M(x)|^p \, dxdy.$$

The estimation of this last integral is very similar to, but slightly simpler than the other one, since the set where $x$ and $y$ are big but $x - \theta y$ is small needs special consideration in the first case, while the set where $x$ and $y$ are big but $x + \theta y$ is small is empty in the second case. Thus we only need to estimate

$$\int_0^{1/2}\int_0^{1/2} |y|^{-p} |D_M(x-\theta y) - D_M(x)|^p \, dxdy.$$

We may also assume that $\theta = 1$, for making the substitution $x' = x$, $y' = \theta y$ gives

$$\theta^{p-1}\int_0^{\theta/2}\int_0^{\theta/2} |y|^{-p} |D_M(x-y) - D_M(x)|^p \, dxdy,$$

which is certainly dominated by

$$\int_0^{1/2}\int_0^{1/2} |y|^{-p} |D_M(x-y) - D_M(x)|^p \, dxdy,$$

uniformly for $\theta \in (0, 1]$.

Partition $[0, 1/2]^2$ into five parts, $R, S, T, U, V$. A picture is useful: $U = [0, \frac{2}{N}] \times [0, \frac{1}{N}]$ is the lower left corner, $V$ is the remainder of the bottom strip of thickness $N^{-1}$, $S$ is the remainder of the left strip of thickness $N^{-1}$. The remaining square with lower left corner $(N^{-1}, N^{-1})$ and upper right corner $(1/2, 1/2)$ is made up of $T$, a thin diagonal strip whose upper boundary lies on the line $y - x = N^{-1}$ and whose lower boundary lies on the line $y - x = -N^{-1}$and of $R$, a union of an upper left triangle and a lower right triangle.

First, since $S$ is where $N^{-1} < y$ and $0 < x < N^{-1}$, then

$$\int_S \ll \int_{N^{-1}}^{1/2} \frac{1}{y^p}\left(\int_0^{N^{-1}} (N+N)^p \, dx\right) dy$$
$$\ll \left(y^{-p+1}|_{1/2}^{N^{-1}}\right) N^p N^{-1} = O\left(N^{2p-2}\right).$$

Second, $R$ is where all of $x$, $y$, and $|x-y|$ are $> N^{-1}$ so that

$$\int_R \ll \int_{N^{-1}}^{1/2} \frac{1}{y^p}\left(\int_{N^{-1}}^{1/2}\left(\frac{1}{|x-y|} + \frac{1}{x}\right)^p dx\right) dy$$
$$\ll \int_{N^{-1}}^{1/2} \frac{1}{y^p}\left(\int_{N^{-1}}^{1/2} \frac{1}{t^p} dt\right) dy + \int_{N^{-1}}^{1/2} \frac{1}{y^p}\left(\int_{N^{-1}}^{1/2} \frac{1}{x^p} dx\right) dy$$

$$= 2\left(y^{-p+1}|_{1/2}^{N^{-1}}\right)\left(x^{-p+1}|_{1/2}^{N^{-1}}\right) = O\left(N^{2p-2}\right).$$

Notice that here and elsewhere we use $\|f+g\|_p^p \le 2^p(\|f\|_p^p + \|g\|_p^p)$, which we sometimes write in the form $\|f+g\|_p^p \ll \|f\|_p^p + \|g\|_p^p$.

Third, $T$ is where $|x-y| < N^{-1}$, but $x > N^{-1}$ and $y > N^{-1}$ so that

$$\begin{aligned}
\int_T &\ll \int_T \frac{1}{y^p}\left(\int \left(N+\frac{1}{x}\right)^p dx\right) dy \\
&\ll \int_{N^{-1}}^{1/2} \frac{1}{y^p}\left(\int_{L_y} N^p dt\right) dy + \int_{N^{-1}}^{1/2} \frac{1}{y^p}\left(\int_{L_y} \frac{1}{x^p} dx\right) dy \\
&= 2\left(y^{-p+1}|_{1/2}^{N^{-1}}\right)\left(N^{p-1}+N^{p-1}\right) = O\left(N^{2p-2}\right),
\end{aligned}$$

where for the first integral we use $|L_y| = |\{(x,y)\in T\}| = O(N^{-1})$ and for the second one we use $\inf\{x : (x,y) \in L_y\} > N^{-1}$.

Next, $U$ is where $x < 2N^{-1}$, $y \in (0, N^{-1})$, so the estimate

$$|D_M(x-y) - D_M(x)| \le \sup\left|D_M'\right| |y|$$

yields

$$\begin{aligned}
\int_U &\ll \int_0^{N^{-1}} \frac{1}{y^p}\left(\int_0^{2N^{-1}} \left(N^2 y\right)^p dx\right) dy \\
&= O\left(N^{2p}N^{-1}N^{-1}\right) = O\left(N^{2p-2}\right).
\end{aligned}$$

What remains is $V$, where $x > 2N^{-1}$, $y < N^{-1}$ and we must estimate

$$\int_V = \int_0^{N^{-1}} \left(\int_{2N^{-1}}^{1/2} \frac{1}{y^p} |D_M(x-y) - D_M(x)|^p dx\right) dy.$$

Apply the mean value inequality $|D_M(x-y) - D_M(x)| \le \sup|D_M'||y|$ and estimate (D4) to get

$$\begin{aligned}
\int_V &= \int_0^{N^{-1}} \left(\int_{2N^{-1}}^{1/2} \frac{1}{y^p} |D_M(x-y) - D_M(x)|^p dx\right) dy \\
&\ll \int_0^{N^{-1}} \left(\int_{2N^{-1}}^{1/2} \frac{1}{y^p} \frac{M^p y^p}{\inf_{0<t<y} |x-t|^p} dx\right) dy \\
&\ll M^p \int_0^{N^{-1}} dy \left(\int_{2N^{-1}}^{\infty} \frac{1}{x^p} dx\right) \ll N^p N^{-1} N^{p-1} = N^{2p-2},
\end{aligned}$$

since $x > 2N^{-1}$ and $y < N^{-1}$ implies that

$$\inf_{0<t<y} |x-t| = x - y \ge \frac{x}{2} \gg x.$$

This completes the proof that $I_B$ is $O(N^{2p-2})$. So we conclude that

$$I_1 = O_p\left(N^{2p-2}\right). \tag{5}$$

It remains to show that

$$I_2 = \int_{\mathbb{T}^2} |J_2(x)|^p\, dx = O\left(N^{2p-2}\right).$$

Since

$$|J_2(x)| = |e((\theta N+1)x_2)| \left| \sum_{n_1=0}^{M} e(n_1(x_1-\theta x_2)) \frac{e(-\{\theta(N-n_1)\}x_2)-1}{e(x_2)-1} \right|$$
$$= \left| \sum_{n_1=0}^{M} e(n_1(x_1-\theta x_2)) \frac{e(-\{\theta(N-n_1)\}x_2)-1}{e(x_2)-1} \right|,$$

$$I_2 \ll \int_{\mathbb{T}^2} \frac{1}{|x_2|^p} \left| \sum_{n_1=0}^{M} e(n(x_1-\theta x_2))(e(-\{\theta(N-n)\}x_2)-1) \right|^p dx$$
$$\le \int_{\mathbb{T}^2} \frac{1}{|x_2|^p} \left| \sum_{n=0}^{M} e(n(x_1-\theta x_2)) \sum_{s=1}^{\infty} \frac{(-2\pi i x_2)^s (\{\theta(N-n)\})^s}{s!} \right|^p dx$$
$$\le \int_{\mathbb{T}^2} \left( \sum_{s=1}^{\infty} \frac{(2\pi)^s}{s!} \left| \sum_{n=0}^{M} e(n(x_1-\theta x_2))(\{\theta(N-n)\})^s \right| \right)^p dx$$
$$= \int_{\mathbb{T}^2} \left( \sum_{s=1}^{\infty} \frac{(2\pi)^s}{s!} \left| \sum_{n=0}^{M} e(nx_1)(\{\theta(N-n)\})^s \right| \right)^p dx.$$

Denoting $(\int_{\mathbb{T}^2} |f|^p)^{1/p}$ by $\|f\|_p$ the last inequality is

$$I_2^{1/p} \ll \left\| \sum_{s=1}^{\infty} \frac{(2\pi)^s}{s!} \left| \sum_{n=0}^{M} e(nx_1)(\{\theta(N-n)\})^s \right| \right\|_p$$

so by Minkowski's inequality,

$$I_2^{1/p} \ll \sum_{s=1}^{\infty} \frac{(2\pi)^s}{s!} \left\| \sum_{n=0}^{M} e(nx_1)(\{\theta(N-n)\})^s \right\|_p$$

which can be rewritten as

$$I_2^{1/p} \ll \sum_{s=1}^{\infty} \frac{(2\pi)^s}{s!} \left( \int_{\mathbb{T}^2} \left| \sum_{n=0}^{M} e(nx_1)(\{\theta(N-n)\})^s \right|^p dx \right)^{1/p}$$

$$= \sum_{s=1}^{\infty} \frac{(2\pi)^s}{s!} \left( \int_{\mathbb{T}} \left| \sum_{n=0}^{M} e(nx) \left( \{\theta (N-n)\} \right)^s \right|^p dx \right)^{1/p}$$

$$= \sum_{s=1}^{\infty} \frac{(2\pi)^s}{s!} \left( H_{p,\theta,N}(s) \right)^{1/p}.$$

In the penultimate step, we used $\int_{\mathbb{T}^2} |f(x_1)| dx_1 dx_2 = \int_0^1 dx_2 \int_0^1 |f(x_1)| dx_1 = 1 \cdot \int_0^1 |f(x)| dx$. We now have to find a good estimate for $H_{p,\theta,N}(s)$.

If we knew that

$$H_{\theta,N}(s) \ll N^{p-1} \ln^p s + N^{p-1} \ln^p N + \frac{s^p}{N^{3p}},$$

then we would have

$$\begin{aligned} I_2^{1/p} &\ll \sum_{s=1}^{\infty} \frac{(2\pi)^s}{s!} \left( N^{1-1/p} \ln s + N^{1-1/p} \ln N + \frac{s}{N^3} \right) \\ &= \left( \sum_{s=1}^{\infty} \frac{(2\pi)^s \ln s}{s!} \right) N^{1-1/p} + \left( \sum_{s=1}^{\infty} \frac{(2\pi)^s}{s!} \right) N^{1-1/p} \ln N \\ &\quad + \left( \sum_{s=1}^{\infty} \frac{(2\pi)^s}{(s-1)!} \right) N^{-3} \\ &= O_p \left( N^{1-1/p} \ln N \right), \qquad (6) \\ I_2 &\ll O_p \left( N^{p-1} \ln^p N \right) \qquad (7) \end{aligned}$$

since the infinite sums are constants for each $p$. Finally estimates (5) and (7) prove the theorem. □

**Lemma 3**

$$H_{p,\theta,N}(s) \ll N^{p-1} \ln^p s + N^{p-1} \ln^p N + \frac{s^p}{N^{3p}}.$$

*Proof* Let us set $\epsilon = M^{-4}$ and

$$\varphi_\epsilon(u) = \begin{cases} u^s & 0 \le u \le 1-\epsilon, \\ \frac{1}{\epsilon} (1-\epsilon)^s (1-u) & 1-\epsilon \le u \le 1, \end{cases}$$

and extend it to be 1-periodic. Expand $\varphi$ into its Fourier series:

$$\varphi_\epsilon(u) = \sum \hat{\varphi}_\epsilon(\nu) e(\nu x).$$

The Fourier coefficients satisfy

(1) $|\hat{\varphi}_\epsilon(\nu)| \le 1,\ \nu \in \mathbb{Z}$,
(2) $|\hat{\varphi}_\epsilon(\nu)| \le |\nu|^{-1}\operatorname{Var}\varphi_\epsilon \ll |\nu|^{-1},\ \nu \ne 0$,
(3) $|\hat{\varphi}_\epsilon(\nu)| \le |\nu|^{-2}\operatorname{Var}\varphi'_\epsilon \ll s\epsilon^{-1}|\nu|^{-2},\ \nu \ne 0$.

To see (1), notice that $\varphi_\epsilon$ is nonnegative and continuous and has its maximum value on $[0, 1]$ at $x = 1 - \epsilon$ and that value is $(1-\epsilon)^s$ which is bounded by 1. To see (2), notice that $\varphi_\epsilon$ is monotone increasing on $[0, 1-\epsilon]$ and monotone (linearly) decreasing on $[1-\epsilon, 1]$ so its total variation is bounded by (2). To see (3), notice that

$$\varphi'_\epsilon(u) = \begin{cases} su^{s-1} & 0 < u < 1-\epsilon, \\ -\frac{1}{\epsilon}(1-\epsilon)^s & 1-\epsilon < u < 1, \end{cases}$$

so the total variation of $\varphi'_\epsilon$ is the rise of $s(1-\epsilon)^{s-1}$ from $0^+$ to $(1-\epsilon)^-$ plus the jump of $s(1-\epsilon)^{s-1} - (-\frac{1}{\epsilon}(1-\epsilon)^s)$ at $x = 1-\epsilon$ plus the jump of $\frac{1}{\epsilon}(1-\epsilon)^s$ at $x = 1$. All three of these quantities are $\ll s\epsilon^{-1}$.

Further set $A = \{n \in \mathbb{Z} \cap [0, M] : \{\theta(N-n)\} \le 1-\epsilon\}$ and let $\bar{A} = \mathbb{Z} \cap [0, M] \setminus A$ be the complement of $A$ in $\{0, 1, 2, \ldots, M\}$. Then

$$\begin{aligned} H_{p,\theta,N}(s) &= \int_{\mathbb{T}} \left| \sum_{n=0}^{M} e(nx)\left(\{\theta(N-n)\}\right)^s \right|^p dx \\ &\ll \int_{\mathbb{T}} \left| \sum_{n\in A} e(nx)\left(\{\theta(N-n)\}\right)^s \right|^p dx \\ &\quad + \int_{\mathbb{T}} \left| \sum_{n\in \bar{A}} e(nx)\left(\{\theta(N-n)\}\right)^s \right|^p dx \\ &= G_1 + G_2. \end{aligned}$$

Since $\varphi_\epsilon(\theta(N-n)) = \{(\theta(N-n))\}^s$ for $\{(\theta(N-n))\} \le 1-\epsilon$, we have

$$\begin{aligned} G_1 &= \int_{\mathbb{T}} \left| \sum_{n\in A} \varphi_\epsilon(\theta(N-n))\, e(nx) \right|^p dx, \\ G_1^{1/p} &= \left\| \sum_{n\in A} \varphi_\epsilon(\theta(N-n))\, e(nx) \right\|_p \\ &= \left\| \sum_{n\in A} \sum_{\nu\in\mathbb{Z}} \hat{\varphi}_\epsilon(\nu)\, e(\nu\theta(N-n))\, e(nx) \right\|_p \\ &\le \sum_{\nu\in Z} \left|\hat{\varphi}_\epsilon(\nu)\right| \left\| e(\nu\theta N) \sum_{n\in A} e(n(x-\nu\theta)) \right\|_p . \end{aligned}$$

But observing $|e(\nu\theta N)| = 1$, making the variable change $x' = x - \nu\theta$, noting $\bar{A} = \{0, 1, \ldots, M\} \setminus A$, and applying Minkowski's inequality gives

$$\begin{aligned}
&\left\| e(\nu\theta N) \sum_{n\in A} e(n(x-\nu\theta)) \right\|_p \\
&\quad = \left\| \sum_{n\in A} e(n(x-\nu\theta)) \right\|_p = \left\| \sum_{n\in A} e(nx) \right\|_p \\
&\quad = \left\| \sum_{n=0}^{M} e(nx) - \sum_{n\in\bar{A}} e(nx) \right\|_p \le \left\| \sum_{n=0}^{M} e(nx) \right\|_p + \left\| \sum_{n\in\bar{A}} e(nx) \right\|_p ,
\end{aligned}$$

so

$$G_1 \le 2^p \left( \sum_{\nu\in Z} \left| \hat{\varphi}_\epsilon(\nu) \right| \right)^p \left( \int_{\mathbb{T}} \left| \sum_{n=0}^{M} e(nx) \right|^p dx + \int_{\mathbb{T}} \left| \sum_{n\in\bar{A}} e(nx) \right|^p dx \right).$$

We use the estimates $\int |D_M|^p = O(N^{p-1}t)$ and

$$\begin{aligned}
\sum_{\nu=-\infty}^{\infty} \left| \hat{\varphi}_\epsilon(\nu) \right| &= \left| \hat{\varphi}_\epsilon(0) \right| + \sum_{1\le|\nu|\le\frac{s}{\epsilon}} + \sum_{|\nu|>\frac{s}{\epsilon}} \\
&\ll 1 + \sum_{1\le|\nu|\le\frac{s}{\epsilon}} \frac{1}{|\nu|} + \sum_{|\nu|>\frac{s}{\epsilon}} \frac{1}{|\nu|^2} \\
&\ll 1 + \ln\left(\frac{s}{\epsilon}\right) + 1/\left(\frac{s}{\epsilon}\right) \\
&= 1 + \ln(sM^4) + \frac{1}{sM^4} \\
&\ll \ln(sN),
\end{aligned}$$

and our estimate for $G_1$ becomes

$$G_1 \ll \ln(sN) \left( N^{p-1} + \int_{\mathbb{T}} \left| \sum_{n\in\bar{A}} e(nx) \right|^p dx \right). \tag{8}$$

Turning to $G_2$, for $n \in \bar{A}$, $\{(\theta(N-n))\} > 1-\epsilon$, so that

$$\{(\theta(N-n))\}^s > (1-\epsilon)^s$$

and

$$\begin{aligned}
\left| 1 - \{(\theta(N-n))\}^s \right| &= 1 - \{(\theta(N-n))\}^s \\
&< 1 - (1-\epsilon)^s \\
&\le s\epsilon.
\end{aligned}$$

Therefore

$$\begin{aligned} G_2 &= \int_{\mathbb{T}} \left| \sum_{n\in\bar{A}} e(nx)\left(1+\{\theta(N-n)\}^s-1\right)\right|^p dx \\ &= \int_{\mathbb{T}} \left| \sum_{n\in\bar{A}} e(nx) + \sum_{n\in\bar{A}} e(nx)\left(\{\theta(N-n)\}^s-1\right)\right|^p dx \\ &\le 2^p \left( \int_{\mathbb{T}} \left| \sum_{n\in\bar{A}} e(nx)\right|^p dx + ((M+1)\epsilon s)^p \right). \end{aligned}$$

But $\epsilon = M^{-4}$ so

$$G_2 \ll \int_{\mathbb{T}} \left| \sum_{n\in\bar{A}} e(nx)\right| dx + \frac{s^p}{N^{3p}}. \tag{9}$$

Our final assertion is that

$$\int_{\mathbb{T}} \left| \sum_{n\in\bar{A}} e(nx)\right|^p dx \ll N^{p-1}. \tag{10}$$

If (10) holds, then from (8)

$$\begin{aligned} H_{p,\theta,N}(s) &\le G_1 + G_2 \\ &\ll \ln^p(sN)\left(N^{p-1}+N^{p-1}\right) + \frac{s^p}{N^{3p}} + N^{p-1} \\ &\ll N^{p-1}\ln^p s + N^{p-1}\ln^p N + \frac{s^p}{N^{3p}}, \end{aligned}$$

since $\ln^p sN = (\ln s + \ln N)^p \ll \ln^p s + \ln^p N$, which finishes the lemma and hence the theorem. So all that remains is the verification of relation (10).

The idea of the rest of the proof is to show that the elements of $\bar{A}$ lie in an arithmetic progression (or possibly in 2 of them), from which (10) is obvious. Explicitly, if $\bar{A} = \{a+b, 2a+b, \dots, ka+b\} \cup \{c+d, 2c+d, \dots, kc+d\}$, then

$$\begin{aligned} \int_{\mathbb{T}} \left| \sum_{n\in\bar{A}} e(nx)\right|^p dx &\le 2^p \int_{\mathbb{T}} \left| \sum_{j=1}^{k} e((ja+b)x)\right|^p dx \\ &\quad + 2^p \int_{\mathbb{T}} \left| \sum_{j=1}^{\ell} e((jc+d)x)\right|^p dx \\ &= 2^p \int_0^1 \left| \sum_{j=1}^{k} e((ja)x)\right|^p dx + 2^p \int_0^1 \left| \sum_{j=1}^{\ell} e((jc)x)\right|^p dx \end{aligned}$$

$$= 2^p \frac{1}{a}\int_0^a \left|\sum_{j=1}^{k} e(jy)\right|^p dy + 2^p \frac{1}{c}\int_0^c \left|\sum_{j=1}^{\ell} e(jy)\right|^p dy$$
$$= 2^p \int_0^1 \left|\sum_{j=1}^{k} e(jy)\right|^p dy + 2^p \int_0^1 \left|\sum_{j=1}^{\ell} e(jy)\right|^p dy$$
$$= O_p(N^{p-1}).$$ □

Thus it only remains to prove the following lemma.

**Lemma 4** *Let $N$ be a positive real number and let $\theta \in (0,1]$. Let $M = \lfloor N \rfloor$ and $\epsilon = M^{-4}$. Then $\bar{A} = \{n \in \{0,1,\dots,M\} : 1-\epsilon < \{\theta(N-n)\} < 1\}$ is an arithmetic progression or the union of two arithmetic progressions.*

*Proof* We distinguish two cases, $\theta$ irrational and $\theta$ rational.

Case 1. $\theta$ irrational.

Let $\{Q_k : k = 1,2,\dots\}$ be the sequence of denominators of the convergents of the continued fraction expansion of $\theta$. Define the positive integer $k_0$ by

$$Q_{k_0} \le M < Q_{k_0+1}.$$

Let $n_1, n_2 \in \bar{A}$. Then from $\{\theta(N-n_1)\} > 1-\epsilon$ and $\{\theta(N-n_2)\} > 1-\epsilon$ we get

$$\begin{aligned}\|\theta(n_1-n_2)\| &= \min_{\ell\in\mathbb{Z}} |\theta(n_1-n_2)-\ell| \\ &\le |(\theta(N-n_2)-\theta(N-n_1)) - (k_2-k_1)| \\ &= |\{\theta(N-n_2)\} - \{\theta(N-n_1)\}|\end{aligned}$$

where $\{\theta(N-n_i)\} = \theta(N-n_i) - k_i$ for integers $k_i$. Thus

$$\|\theta(n_1-n_2)\| < \epsilon = \frac{1}{M^4}, \tag{11}$$

where $\|\cdot\|$ denotes the distance to the nearest integer.

Since $|n_1-n_2| \le M$, there is an $m$, $1 \le m \le M$ such that

$$\|\theta m\| < \epsilon. \tag{12}$$

From elementary number theory we will need the following three facts.

Let $P_k/Q_k$ and $P_{k+1}/Q_{k+1}$ be successive convergents of $\theta$. Then

$$\text{If } m < Q_{k+1}, \quad \text{then for all } \ell \in \mathbb{Z}, \quad |\theta m - \ell| \ge |\theta Q_k - P_k|, \tag{13}$$

$$|\theta Q_k - P_k| > \frac{1}{Q_k + Q_{k+1}}, \tag{14}$$

$$\left|\theta - \frac{P_k}{Q_k}\right| < \frac{1}{Q_k Q_{k+1}}. \tag{15}$$

See Theorem 7.13 of [7] for the first fact, Theorem 13 in Khinchin's book [4] for the second, and Theorem 7.11 of [7] for the third.

Next we estimate $\|\theta m\|$ from below. Since $m \le M < Q_{k+1}$, from facts (13) and (14) we get

$$\|\theta m\| = \min_{\ell\in\mathbb{Z}} |\theta m - \ell| > \frac{1}{Q_{k_0} + Q_{k_0+1}}. \tag{16}$$

Putting (16) together with (12), gives

$$\frac{1}{M^4} = \epsilon > \frac{1}{Q_{k_0} + Q_{k_0+1}} \ge \frac{1}{M + Q_{k_0+1}}.$$

This is only possible if $Q_{k_0+1}$ is $O(M^4)$; explicitly, from this we get

$$\begin{aligned} M + Q_{k_0+1} &> M^4, \\ Q_{k_0+1} &> M^4 - M, \end{aligned}$$

so that as soon as $M \ge 2$,

$$Q_{k_0+1} \ge \frac{1}{2} M^4. \tag{17}$$

We show that if $n_1, n_2 \in \bar{A}$, then $n_1 - n_2 \equiv 0 \pmod{Q_{k_0}}$. Indeed let $n_2 - n_1 = q Q_{k_0} + r$, where $0 < r < Q_{k_0}$. Then

$$\begin{aligned} \|\theta\,(n_1 - n_2)\| &= \left\|\theta\left(q Q_{k_0} + r\right)\right\| \\ &\ge \|\theta r\| - \left\|\theta\left(q Q_{k_0}\right)\right\| \\ &\ge \|\theta r\| - q\left\|\theta Q_{k_0}\right\|. \end{aligned} \tag{18}$$

For the last step, note that for any norm and any integer $q$,

$$\begin{aligned} \|qa\| &= \|a + \cdots + a\| \\ &\le \|a\| + \cdots + \|a\| = q\,\|a\|. \end{aligned}$$

We also have

$$\|\theta r\| \ge \frac{1}{2Q_{k_0}}; \tag{19}$$

since applying facts (13) and (14) with $k = k_0 - 1$ gives

$$\begin{aligned} \|\theta r\| &= \min_{\ell\in\mathbb{Z}} |\theta r - \ell| \\ &> \frac{1}{Q_{k_0-1} + Q_{k_0}} \\ &> \frac{1}{Q_{k_0} + Q_{k_0}}. \end{aligned}$$

Also

$$\begin{aligned}\left\|\theta Q_{k_0}\right\| &= \left\| Q_{k_0}\frac{P_{k_0}}{Q_{k_0}} + \left(\theta Q_{k_0} - Q_{k_0}\frac{P_{k_0}}{Q_{k_0}}\right)\right\| \\ &= \left\| P_{k_0} + \left(\theta - \frac{P_{k_0}}{Q_{k_0}}\right) Q_{k_0}\right\| \\ &= \left\| \left(\theta - \frac{P_{k_0}}{Q_{k_0}}\right) Q_{k_0}\right\| \\ &\le Q_{k_0}\left\| \left(\theta - \frac{P_{k_0}}{Q_{k_0}}\right)\right\| \\ &\le Q_{k_0}\left| \left(\theta - \frac{P_{k_0}}{Q_{k_0}}\right)\right| \\ &< Q_{k_0}\frac{1}{Q_{k_0}Q_{k_0+1}} = \frac{1}{Q_{k_0+1}}.\end{aligned}$$

Taking this and inequality (19) into account,

$$\|\theta r\| - q\left\|\theta Q_{k_0}\right\| \ge \frac{1}{2Q_{k_0}} - q\frac{1}{Q_{k_0+1}}.$$

Combining this with (18) we have

$$\|\theta(n_1 - n_2)\| \ge \frac{1}{2Q_{k_0}} - q\frac{1}{Q_{k_0+1}}.$$

Taking $Q_{k_0} \le M$, $q \le M$ and (17) into account, from this we get

$$\begin{aligned}\|\theta(n_1 - n_2)\| &\ge \frac{1}{2M} - M\frac{1}{1/2M^4} \\ &= \frac{1/2M^3 - 2M}{M^4}.\end{aligned}$$

As soon as $M \ge 3$, this is contrary to

$$\|\theta(n_1 - n_2)\| < \epsilon = \frac{1}{M^4}.$$

Consequently all the $n_i$ are congruent mod $Q_{k_0}$ and there is a single $r$, $0 \le r < Q_{k_0}$ so that to every $n \in \bar{A}$ there corresponds a $q$ so that

$$n = qQ_{k_0} + r.$$

Let us consider the sequence $x_q$, $0 \le q \le t = \lfloor \frac{M-r}{Q_{k_0}} \rfloor$, where for each integer $q$, $x_q = \{\theta(N - (qQ_{k_0} + r))\}$ is on the circle of circumference 1. If $q_1 \neq q_2$, then $\theta$ irrational implies that $\theta(N - (q_2 Q_{k_0} + r)) - \theta(N - (q_1 Q_{k_0} + r)) = \theta(q_1 - q_2)Q_{k_0}$ is not an integer, so that $x_{q_1} \neq x_{q_2}$.

For appropriate integers $j$ and $k$,

$$\begin{aligned}\left|x_q - x_{q+1}\right| &= \left|\left(\theta\left(N-\left(qQ_{k_0}+r\right)\right)-j\right)-\theta\left(N-\left((q+1)Q_{k_0}-k\right)\right)\right| \\ &= \left|\theta Q_{k_0}-(j-k)\right| \\ &\geq \min_{\ell\in\mathbb{Z}}\left|\theta Q_{k_0}-\ell\right| = \left\|\theta Q_{k_0}\right\|.\end{aligned}$$

But we must actually have equality here:

$$\text{for all } q=0,1,\dots,t-1, \quad \left|x_q - x_{q+1}\right| = \left\|\theta Q_{k_0}\right\|,$$

since the left and right sides are both in (0, 1) and can only differ by an integer.

Thus, starting from the point $x_0$ and passing to the point $x_1$, and so on, either in the clockwise or the counterclockwise direction the points $x_0, x_1, \dots, x_t$ wind regularly around the unit circumference with constant step length $\|\theta Q_{k_0}\|$. Let us observe that no $x_q$ falls between $x_k$ and $x_{k+1}$ in this passage. Indeed, if $x_{k+1} < x_q < x_k$, on the one hand, $|x_q - x_k| < \|\theta Q_{k_0}\| = \min_{\ell\in\mathbb{Z}}|\theta Q_{k_0}-\ell| \leq |\theta Q_{k_0} - P_{k_0}|$. On the other hand since $Q_{k_0}|k-q| < Q_{k_0+1}$, by fact (13),

$$\begin{aligned}\left|\theta Q_{k_0} - P_{k_0}\right| &\leq \left|\theta Q_{k_0}(k-q) - \left(\lfloor x_q\rfloor - \lfloor x_k\rfloor\right)\right| \\ &= \left|\left(\theta\left(N-\left(qQ_{k_0}+r\right)\right)-\lfloor x_q\rfloor\right)-\left(\theta\left(N-\left(kQ_{k_0}+r\right)\right)-\lfloor x_k\rfloor\right)\right| \\ &= \left|x_q - x_k\right|,\end{aligned}$$

which is a contradiction. Consequently, the set $S = \{q : x_q \in I\}$ where $I$ is an interval on the unit circle is either a set of consecutive integers in which case $\{x_q \in I\}$ is an arithmetic progression; or else $S$ starts and ends in $I$ so that $S = \{0, 1, 2, \dots, a\} \cup \{b, b+1, \dots, t-1, t\}$ is a union of two sets of consecutive integers in which case $\{x_q \in I\}$ is a union of two arithmetic progressions. In particular, if $I = (1-\epsilon, 1)$, then $\{x_q \in I\} = \bar{A}$, so that $\bar{A}$ is the union of at most two arithmetic progressions with common difference $Q_{k_0}$.

Case 2. $\theta$ rational.

Let $Q_1 < Q_2 < \cdots < Q_k$ be the denominators of all the convergents of the continued fractions of $\theta$.

If $M < Q_k$ the proof follows as in Case 1 with $k_0 \leq k-1$. If $Q_k \leq M$, then we let $n_1 \in \bar{A}$. Since $\theta = P_k/Q_k$

$$\bar{A} = \left\{n \in \mathbb{Z}\cap[0, M] : \left\{\frac{P_k}{Q_k}(N-n)\right\} > 1-\epsilon\right\}.$$

Since

$$\left\{\frac{P_k}{Q_k}(N-n_1)\right\} = \left\{\frac{P_k}{Q_k}(N-k)\right\}$$

for arbitrary $k \equiv n_1 \pmod{Q_k}$, it follows that $\bar{A}$ contains all $k$ such that $0 \leq k \leq M$ and $k \equiv n_1 \pmod{Q_k}$. The set $\bar{A}$ contains no other elements since if $k$ is not congruent to $n_1 \pmod{Q_k}$, then

$$\left|\left\{\frac{P_k}{Q_k}(N-n_1)\right\} - \left\{\frac{P_k}{Q_k}(N-k)\right\}\right| \geq \left\|\frac{P_k}{Q_k}(n_1-k)\right\| \geq \frac{1}{Q_k} > \epsilon,$$

i.e., $\{\frac{P_k}{Q_k}(N-n_1)\}$ and $\{\frac{P_k}{Q_k}(N-k)\}$ cannot both belong to an interval of length $\epsilon$ and therefore

$$\left\{\frac{P_k}{Q_k}(N-k)\right\} < 1-\epsilon.$$

Consequently, $\bar{A}$ is an arithmetic progression with common difference $Q_k$ and the lemma is proved. □

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