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REGULAR ARTICLE

Stock prices and the real economy: power law versus exponential distributions

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Abstract This paper explores the relationship between the stock prices and the real economy. The standard approach – the so called consumption-based asset pricing model – attempts to explain it based on the assumption of the representative agent. In this paper, we argue that the representative agent assumption is fundamentally flawed. Drawing on the recent advancement of *"econophysics"* on financial markets See Mantegna and Stanley (An Introduction to econophysics: correlations and complexity in finance, 2000) for the introduction to econophysics, we argue that in contrast to the neoclassical view, there is in fact a wedge between financial markets, the stock prices in particular, and the real economy.

1 Introduction

The stock prices depend necessarily on the real economy. Their 'correct' prices or the fundamental values are the discounted present values of a stream of future dividends/profits. Since business activities, profits in particular, are significantly affected by the state of the real economy, the stock prices are also affected by the real economy. More precisely, in the standard neoclassical theory, stock prices are simultaneously determined with all the supplies and demands in general equilibrium (Diamond 1967). Thus, just like production and consumption, the stock prices depend ultimately on preferences and technologies.

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However, there is a long tradition in economics, which questions whether the stock prices are really determined in the way stated above. Many believe that the "bubbles" are possible in the market. And whether or not they are "bubbles," extraordinary changes in the stock prices (either up or down) *by themselves* may do harm to the real economy ; They are not a mere mirror image of the real economy. In history, depressions were often accompanied by falls in the stock prices. As early as in the 19th century, economists were talking about "financial crises." More recently, Minsky (1957, 1975) highlighted the importance of the stock prices in the macroeconomy, and advanced the 'financial accelerator' thesis. It was revived in the 1990s, and bore the vast literature. Today, most central banks closely monitor asset prices in the conduct of monetary policy.

The crucial problem is whether the stock prices are always equal to their fundamental values. Shiller (1981) in his seminal work performed the ingenious "variance-bound tests" on this issue, and drew the following conclusion:

> "We have seen that measures of stock price volatility over the past century appear to be far too high — five to thirteen times too high — to be attributed to new information about future real dividends if uncertainty about future dividends is measured by the sample standard deviations of real dividends around their long-run exponential growth path. (Shiller (1981), p. 433)"

Naturally, Shiller's seminal work[1] spawned the debate over alleged excess volatility of stock price. Rather than accepting that stock prices are too volatile to be consistent with the standard theory, a majority of economists have attempted to reconcile the alleged volatility with efficiency or "rationality" of market.

One way to explain volatility of stock prices is to allow significant changes in the discount rate or the required return on stocks. In fact, in the neoclassical macroeconomic theory, the following relationship between the rate of change in consumption, C, and the return on capital, r must hold in equilibrium (see, for example, Blanchard and Fischer (1989)):

$$-\left[\frac{u''(C)C}{u'(C)}\right]\left(\frac{\dot{C}}{C}\right) = \frac{1}{\eta(C)}\left(\frac{\dot{C}}{C}\right) = r - \delta. \tag{1}$$

Here, the elasticity of intertemporal substitution η is defined as

$$\frac{1}{\eta(C)} = -\frac{u''(C)C}{u'(C)} > 0.$$

In general, η depends on the level of consumption, C. Equation (1) says that the rate of change in consumption over time is determined by η and the difference

[1] The variance bound test for the volatility of stock prices was also performed by LeRoy and Porter (1981).

between the rate of return on capital, r and the consumer's subjective discount rate, δ. This equation, called *the Euler equation*, is derived as the necessary condition of the representative consumer's maximization of the Ramsey utility sum.

The return on 'capital,' r in equation (1) is the return on capital equity or stocks, which consists of the expected capital gains/losses and dividends. Thus, according to the neoclassical macroeconomics, the return on stocks must be consistent with the rate of change in consumption over time in such a way that Eq. (1) holds.

Now, the results of the tests of Shiller (1981) and LeRoy and Porter (1981) imply that the volatility of stock prices must come from the volatility of the discount rate or the return on capital, r, rather than that of dividends. And yet, consumption is not volatile. If anything, it is *less* volatile than dividends or profits. Thus, given Eq. (1), the volatility of stock prices or their rate of return r must be explained ultimately by sizable fluctuations of the elasticity of intertemporal substitution η which depends on consumption. Consequently, on the representative agent assumption, researchers focus on the "shape" of the utility function in accounting for the volatility of stock prices (Grossman and Shiller 1981). It is not an easy task, however, to reconcile the theory with the observed data if we make a simple assumption for the elasticity of intertemproral substitution, η; η must change a lot despite of the fact that changes in consumption are small.

A slightly different assumption favored by theorists in this game is that the utility, and therefore, this elasticity η depend not on the current level of consumption C_t but on its deviation from the "habit" level, $\hat{C}_t$, namely, $C_t - \hat{C}_t$. By assumption, the habit $\hat{C}_t$ changes much more slowly than consumption C_t itself so that at each moment in time, $\hat{C}_t$ is almost constant. The trick of this alternative assumption is that although C_t does not fall close to zero, $C_t - \hat{C}_t$ can do so as to make the elasticity of intertemporal substitution η, now redefined as

$$\frac{1}{\eta} = -\frac{u''(C-\hat{C})(C-\hat{C})}{u'(C-\hat{C})} > 0, \tag{2}$$

quite volatile. Campbell and Cochrane (1999) is a primary example of such an approach. Though ingenious, the assumption is not entirely persuasive. Why does the consumer's utility become minimal when the level of consumption is equal to the habit level even if it is extremely high? In any case, this is the kind of end point we are led to as long as we keep the representative agent assumption in accounting for the volatility of stock prices.

Meanwhile, Mehra and Prescott (1985) using the representative agent model, presented another problem for asset prices. They considered a simple stochastic Arrow-Debreu model. The model has two assets, one the equity share for which dividends are stochastic, and the other the riskless security. Again, on the representative agent assumption, the "shape" of the utility function and the volatility of consumption play the central role for prices of or returns on two assets. For the reasonable values of η, which may be more appropriately called the relative risk aversion in this stochastic model, and the U.S. historical standard deviation of consumption growth, Mehra and Prescott (1985) calculated the theoretical values of the returns on two assets. The risk premium, namely the difference between the return on the equity share and the return on the riskless security implied by their model, turns out to be mere 0.4%. In fact, the actual risk premium for the U.S. stock (the Standard and Poor 500 Index, 1889–1978) against the short-term security such as the Treasure Bills, is 6%. Thus, the standard model with the representative consumer fails to account for such high risk premium that is actually observed.Mehra and Prescott (1985) posed this result as a puzzle. Since then, a number of authors have attempted to explain this puzzle: See Campbell and Cochrane (1999).

The "puzzles" we have seen are, of course, puzzles conditional on the assumption of the representative-agent. Indeed, Deaton (1992) laughs away the so-called "puzzles" as follows:

> "There is something seriously amiss with the model, sufficiently so that it is quite unsafe to make any inference about intertemporal substitution from representative agent models ... .
>
> The main puzzle is not why these representative agent models do not account for the evidence, but why anyone ever thought that they might, given the absurdity of the aggregation assumptions that they require. While not all of the data can necessarily be reconciled with the microeconomic theory, many of the puzzles evaporate once the representative agent is discarded. (Deaton (1992), p.67, 70)."

We second Deaton's criticism. Having said that, here, we note that the standard analyses all focus on the *variance* or the *second moment* of asset prices or returns; see Cecchetti et al. (2000), for example, and the literature cited therein. As we will see it shortly, a number of empirical studies actually demonstrate that the variance or standard deviation may *not* be a good measure of *risk*. We must consider probability distributions, not just moments. In what follows, we will critically examine the consumption-based asset pricing model, and argue that financial markets, stock prices in particular, and the real economy are, in fact, different creatures.

2 The power-law behavior of stock prices and returns

Toward this goal, we must begin with the story of power-law probability distribution. It may appear too technical at first, but is essential for our understanding the workings of financial markets on one hand, and the real economy on the other. Although economists routinely adopt the normal or the Gaussian distribution, it turns out that it is actually not so generic as they believe. Specifically, *power - law* plays the central role for understanding financial markets.[2]

[2] See chapters 4 and 6 of Sornette (2000), and Mandelbrot and Hudson (2004) for the introduction of power-law distributions.

2.1 Power law

Despite its fundamental importance, power - law is relatively unknown among the main stream economists. We first give its definition.

Definition. *(Power-law Distribution):Stochastic variable x is said to obey a power-law distribution when it is characterized by a probability density function $p(x)$ with power-law tails*

$$p(x) \propto x^{-(1+\alpha)} \quad (\alpha > 0).$$

□

To appreciate the importance of the power-law distribution, we need to compare it with the normal distribution. The probability density function of the standard normal distribution $\phi(x)$ is well known:

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right).$$

Economists like scientists in other disciplines have long believed that the normal distribution is the norm, the deviation from which serves only for curiosity of mathematicians. There are several justifications for this belief. The most important one is, of course, *the central limit theorem*. Random walk model is another.

A version of the central limit theorem states as follows.

The central limit theorem: Suppose that x_i is an identically and independently distributed (i.i.d.) random variable with mean zero and a *finite variance* σ^2. Then the probability density function $f_n(s)$ of the (normalized) sum of x_i, $s_n = \sum_{i=1}^{n} x_i / \left[\sigma\sqrt{n}\right]$ converges to a normal distribution with unit variance, as n becomes large:

$$f_n(s) \to \phi(s) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{s^2}{2}\right) \quad \text{when} n \to \infty.$$

□

See any textbook on probability such as Dixit (1991) for technical details. The point is that the theorem allows *any* distribution for x_i, as long as the second moment exists. The normal distribution can be also seen as a limit of *random walk model.*

The central limit theorem, and the fact that the normal distribution can be seen as a limit of random walk, are well known. These facts suggest strongly that the normal distribution is very generic; We should expect the normal distribution everywhere in nature. However, it turns out that the normal distribution may not be so generic as we might believe. A crucial assumption of the central limit theorem is that the probability distribution of x_i has the finite variance. What happens if the variance or the second moment does not exist?

The normal distribution actually belongs to a group of distribution called *stable distribution*. Stable distribution is a specific type of distribution encountered in the sum of n i.i.d. random variables with the property that it does not change its functional form for different values of n. It is known that the normal distribution is the only stable distribution having all its moments finite. Now, there exists a limit theorem stating that, under certain conditions, the probability density function of a sum of n i.i.d. random variables x_i converges in probability to a stable distribution. Note that the central limit theorem is a special case of this more general limit theorem. When pdf of x_i has a finite variance, it becomes the usual central limit theorem; The limit distribution is the normal distribution. On the other hand, when the variance or the second moment does not exist (namely, it becomes infinite) for the underlying stochastic process, a sum of n i.i.d. random variables converges to a distribution with power-law tails which is also a member of the group of stable distribution.

The random walk which leads us to the normal distribution has been regarded as a very generic model with wide applications. However, it is also restrictive in the sense that the length of a jump of a "ball" is constant. More generally, we can consider a "random walk" with the following probability distribution of the lengths of a jump of a "ball":

$$\begin{array}{rll} \pm a & \text{with probability} & C \\ \pm\lambda a & \text{with probability} & C/M \\ \vdots & & \vdots \\ \pm\lambda^j a & \text{with probability} & C/M^j \\ \vdots & & \vdots \end{array} \qquad (a > 0,\ C > 0,\ \lambda > 1,\ M > 1)$$

In this generalized random walk model, a ball can fly to any point on one-dimensional lattice with power-law probabilities: a small jump is more likely than a big jump. This is a one-dimensional example of *Lévy flight*.[3] Now, this generalized random walk, or the Lévy flight is much "wilder" than the ordinary random walk model, and can lead us to power-law distributions rather than to the normal distribution.[4]

In summary, stable distribution is more general than the normal distribution. In this group of probability distribution, we have strong contenders to the normal distribution, namely power-law distributions.

In addition to the kind of limit distribution, we must also take into account the speed of convergence. The problem can be best illustrated by an example.

[3] The model is not restricted to a one-dimensional model. Mandelbrot (1983) coined the term "Lévy flight" for the generalization of random walks in continuous space.

[4] The reader can usefully refer to Figure 4.7 of Sornette(2000, p.93) to appreciate the point that Lévy flight is much "wilder" than the ordinary random walk. For Lévy flights and power-laws, see also chapter 4 of Paul and Baschnagel (1999).

Consider the *truncated* Lévy flight defined by the following distribution:

$$P(x) = \begin{cases} 0 & \text{for} \quad x > m > 0 \\ cP_{\mathrm{L}}(x) & \text{for} \ -m \leq x \leq m \\ 0 & \text{for} \quad x < -m \end{cases}$$

where $P_{\mathrm{L}}(x)$ is the symmetric Lévy flight explained in the text. Unlike the Lévy flight in which the length of a jump is unbounded, the truncated Lévy flight has a limit ($m > 0$) on the length of a jump. Since the truncated Lévy flight has a finite variance, the probability distribution of the sum of n random variables form this distribution, $P(S_n)$ converges to the normal distribution. The question is how quickly $P(S_n)$ will converge. Obviously, when n is small, the Lévy flight well approximates $P(S_n)$. Thus, there exists a crossover value of n, n^* such that

$$\begin{aligned} &\text{For } n \ll n*, \quad P(S_n) \sim \text{the Lévy flight} \\ &\text{For } n \gg n*, \quad P(S_n) \sim \text{the normal distribution.} \end{aligned}$$

This example illustrates the point that in general, the kind of probability distribution we obtain in practical applications depends on n; see Sect. 8.4 of Mantegna and Stanley (2000) for further details.

In fact, more and more evidences have been gathered to the effect that natural phenomena are characterized by power-law distributions (see, for example, Sornette (2000)]. In economics, empirical size distributions of many variables of interest have been actually known for long to obey power-law distribution. For example, Pareto (1896) found that the distribution of income y was of the following form:

$$N(y > x) \sim x^{-(3/2)}$$

where $N(y > x)$ is the number of people having income x or greater than x. The Pareto distribution is nothing but a particular form of power-law distribution.

More recently, electronic trading in financial markets has enabled us to use a rich high-frequency data with the average time delay between two records being as short as a few seconds. By now, a number of empirical analyses based on such data have amply demonstrated that most financial variables such as changes in stock price or foreign exchange rates are, in fact, characterized by power-law distributions, *not* by the normal distribution.

What is the significance of these results? The significant difference between the normal and power-law distributions shows up in tails of distribution as shown in Fig. 1. Under power - laws, large deviations from the mean have much larger probability (dubbed "*fat tails*") than under the normal distribution. Put it another way, given the normal distribution, some of the big earthquakes which actually occurred would not have reasonably occurred whereas they are quite possible under power - laws. Likewise, under the normal distribution, drops in stock price such as the October 1987 Crash would have insignificant probability whereas under power - laws, the probability becomes significant. Power - laws

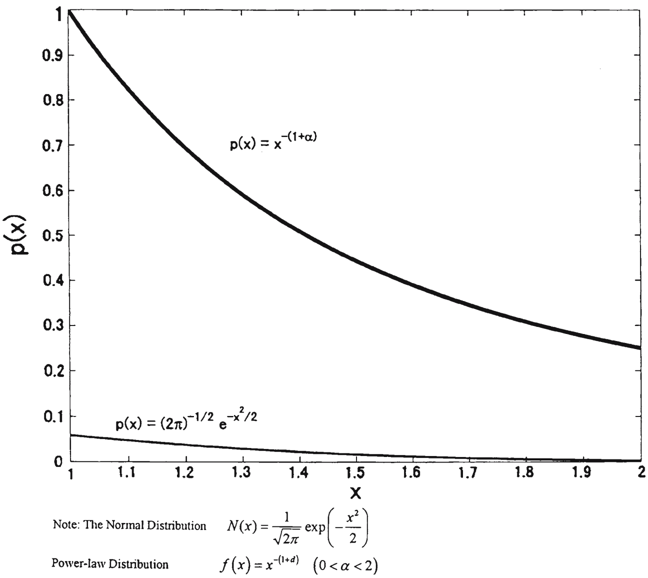

Fig. 1 Tail comparison of standard normal and power - Law ($\alpha = 1$)

have, therefore, important implications for our understanding of financial markets.

2.2 Asset prices

Growing evidences dating back to Mandelbrot (1963) now amply demonstrate that changes in asset prices do not obey the normal distribution but the power-law. For our present purpose, it is enough to cite .

The stylized fact 1. *The probability distribution of changes in stock prices r follows the following power law with the exponent* $\alpha = 3$*:*

$$P(|r| > x) \propto x^{-\alpha}, \qquad \alpha = 3 \tag{3}$$

where, r is defined as follows:

$$r_t = \log P_t - \log P_{t-\Delta t}.$$

Probability density function, $f(r)$ *corresponding to (3) is*

$$f(r) \propto x^{-(\alpha+1)} = x^{-4}. \tag{4}$$

That is, in terms of density function $f(r)$*, r obeys the power law with the exponent* $\alpha + 1 = 4$. □

See also Mandelbrot (1997) and Mantegna and Stanley (2000) for the above stylized fact.

That exponent α is about 3 is the standard result. The value of the exponent has far reaching implications. First of all, when the exponent of the power-law *density function* is 3, the variance or the second moment does not exist. In general, suppose a random variable X has a power-law *density* $f(x)$ in the range $1 \leq x \leq \infty$ with exponent $\mu + 1$:

$$f(x) = x^{-(\mu+1)} \quad (1 \leq x \leq \infty).$$

The nth moment M_n of X is then defined as

$$M_n = \int_1^\infty x^n f(x)\mathrm{d}x = \int_1^\infty x^{-(\mu+1-n)}\mathrm{d}x.$$

Thus, the nth moment of X, M_n exists if and only if

$$\mu + 1 - n > 1 \quad \text{or} \quad \mu > n.$$

In other words, the nth moment of the random variable X does not exist for $\mu \leq n$.

Though it appears that the second moment or variance *does* exist for financial returns [see chapter 9 of Mantegna and Stanley (2000)], it is still a matter of dispute. If the variance does not exist, the standard theory of asset prices faces a serious problem because it rests on the basic assumption that the distribution of returns is normal (Gaussian), and that *risk* can be measured by the variance or standard deviation of the rate of return; See Mandelbrot and Hudson (2004) for very readable and forceful criticism of the standard theory of asset prices and finance.

Whether or not the second moment exists, compared to the normal distribution, power-law distribution have "*fat tails*" meaning that large deviations from the mean have the significant probabilities (Fig. 1). Mandelbrot, a founder of new approach, contrasts two broad classes of probability distributions, one the "mild," the other the "wild." The normal distribution belongs to the "mild" one whereas power-law distributions are "wild." To appreciate the point, following Sornette (2000), we can think of the distribution of height. The probability that someone has twice your height is virtually zero because the distribution of height is normal, and "mild." In contrast, in the case of the distribution of wealth which Pareto (1896) first explored, there is a non-vanishing fraction of the population twice, 10 times or even 100 times as wealthy as you are. The reason is that unlike height, wealth is distributed under the power - law. "Wild" probability distributions are, in fact, found to well approximate the size frequency distributions of a wide range of natural phenomena such as earthquakes, hurricanes, and floods. And now, changes in stock prices have been found to obey a power-law distribution. Dismissing the "equity premium puzzle" mentioned earlier, Mandelbrot draws the important implication for power-law distributions for *risk* as follows:

> "Why is it that stocks, according to the averages, generally reward investors so richly? The data say that, over the long stretch of the twentieth century, stocks provided a massive "premium" return over that of supposedly safer investments, such as U.S. Treasury Bills. Inflation-adjusted estimates of that premium vary, depending on the dates you examine, between 4.1 percent and 8.4 percent. Conventional theory calls this impossible. Only two things, the theory says, could so inflate stock prices: Either the market is so risky that people will not invest otherwise, or people merely *fear* it is too risky and so will not invest otherwise. Now, when studying this, economists typically measure the real market risk by its volatility — quantified by their old friend, the bell-curve standard deviation. ... But these papers miss the point. They assume that the "average" stock-market profit means something to a real person; in fact, it is the extremes of profit or loss that matter most. Just one out-of-the-average year of losing more than a third of capital — as happened with many stocks in 2002 — would justifiably scare even the boldest investors away for a long while. The problem also assumes wrongly that the bell curve is a realistic yardstick for measuring the risk. As I have said often, real prices gyrate much more wildly than the Gaussian standard assume. In this light, there is no puzzle to the equity premium. Real investors know better than the economists. They instinctively realize that the market is very, very risky, riskier than the standard models say. So, to compensate them for taking that risk, they naturally demand and often get a higher return. (Mandelbrot and Hudson (2004), p. 230–231) Mandelbrot and Hudson (2004)"

At this stage, we return to the standard consumption-based theory of asset pricing, namely Eq. (1). According to the basic equation of the consumption-based asset pricing model, the equity return r and the rate of change in consumption must be closely related. Now, the standard empirical analyses which attempt to reconcile the neoclassical theory with the observed data all focus on the *moments* of the relevant variables. However, given the stylized fact that equity returns obey the power - law for which the second moment may not even exist, the standard approach is highly questionable. *The valid approach is to compare the probability distributions – not just the moments – of consumption growth and the stock prices or returns*, namely the left-hand and the right-hand sides of the Euler equation (Eq. 1).

2.3 Growth of real variables

The return on equity obeys the power-law distribution with exponent $\alpha = 3$ (the stylized fact 1 above). What about the rate of change in consumption? Changes in consumption and aggregate income or GDP are similar. Canning et al. (1998) shows that the distribution of the growth rates of GDP, g is exponential.

$$P(g) \sim \exp[-\gamma \mid g \mid]. \tag{5}$$

Stanley et al. (1991) analyzing all U.S. publicly traded manufacturing companies within the years 1975–91 (taken from the Compustat database), drew the conclusion that the distribution of the growth rates of companies is also exponential.

The Stylized Fact 2. *The probability distributions of growth rates of real variables such as GDP, consumption, and the size of firm are exponential. They are fundamentally different from the probability distributions of the asset prices or returns which are power laws.* □

This fact implies that *the standard Euler equation (Eq. 1) based on the representative agent assumption is fundamentally flawed.*

The important problem is then to identify and compare the underlying mechanisms which generate power-law distributions for the returns of financial assets on one hand, and exponential distributions for the real economic activities such as the growth of real GDP on the other.

3 Underlying mechanism: a truncated Lévy flight model

As we explained it in Sect. 2, the random walk leads us to the normal (or Gaussian) distribution. Unlike the standard random walk, *truncated Lévy flight*, explained earlier, depending on parameters, can generate a wide class of probability distributions including power-laws and exponential distribution. In what follows, we will consider a particular model of truncated Lévy flight which nests both power-laws and exponential distributions. The model is an adapted or modified version of Huang and Solomon (2001).

3.1 The real economy

We first consider a model of the real economy. The economy consists of N sectors or units. For the sake of expositional convenience, we call the variable of interest "consumption." It may be "production," and in that case, the aggregate variable is GDP. N sectors or units may be interpreted either as N types of consumers, or as N types of consumption goods. Interpretation of the model can be very flexible.

The aggregate consumption at time t, $C(t)$ is nothing but the sum of the individual consumptions:

$$C(t) = c_1(t) + \cdots + c_N(t), \tag{6}$$

Here, $c_i(t)$ is the consumption of type i good or the ith consumer's consumption. The argument t stands for calendar or real time. We interpret the period from t to $t + 1$ as 1 month, 3 months or 1 year as the case maybe.

We are interested in the growth rate of the *aggregate* consumption $C(t)$ over $[t, t+1]$, $r(t)$

$$r(t) = \frac{C(t+1) - C(t)}{C(t)}. \tag{7}$$

Our goal is to derive the probability distribution of r.

The growth of aggregate (or macro) consumption arises from the aggregation of growths of the N individual (or micro) consumptions. We think of this *micro* growth in the form of a (large) number of *elementary events*. The number of elementary events within a period (namely over $[t, t+1]$) is τ. One elementary event is that a sector, sector i, say, being randomly chosen from the set $\{1, 2, \ldots, N\}$ between t and $t+1$, experiences growth of consumption. We use the term "sector," but it can be any micro unit such as agent, individual or firm. A sector may be chosen either uniformly with probability $1/N$ or with some other probabilities possibly dependent on the size of the sector. If sector i is chosen, $c_i(t)$ grows by a random factor λ:

$$c_i^{'}(t) = \lambda c_i(t). \tag{8}$$

Here, $c_i^{'}(t)$ is defined as c_i immediately after the elementary growth. At this point, no other sector $(j \neq i)$ experiences growth.

For simpler exposition, we assume that

$$\lambda = 1 + g, \qquad \text{for } \forall\ i, t \tag{9}$$

where

$$g = \pm\gamma \qquad (0 < \gamma < 1).$$

Note that this is a particular type of multiplicative process for c_i. As we mention later, the probability distribution of λ does not matter at all.

It is extremely important to keep in mind the difference between t and τ. One is the *calendar time*, t, and the other is τ, the number of *elementary (micro) events* within a given period of time. When there are a total of τ such elementary events undergone by some or all of the N sectors, each sector is most likely to experience more than one elementary events on the average. We denote the resultant growth rate of aggregate consumption as $r(t; \tau)$. That is, $r(t; \tau)$ is the growth rate of $C(t)$ between t and $t+1$ when the number of elementary micro events during the period is τ. Although τ is a random number, we use its expected value and denote it by τ.

With τ random selections out of N sectors in one period, we can write the rate of growth of aggregate consumption as

$$r(t; \tau) = \sum_{i,k} g_{i,k}, \quad (k = 1, \ldots, \tau) \tag{10}$$

where

$$g_{i,k} = \frac{(c_{i,k+1} - c_{i,k})}{C} = \frac{\pm\gamma c_i(t;k)}{C}. \tag{11}$$

Here, $g_{i,k}$ indicates the kth elementary growth that has occured to $c_i(t)$. The total change that defines the growth rate of aggregate consumption $C(t)$ is the sum of these elementary growths that occured to $c_1, \ldots, c_N$. The total number of the elementary events that have occurred is equal to τ.

In this model, the size of a jump of a micro unit is constant (Eq. 9). Thus, the micro behavior is described by the ordinary random walk. However, such micro growths occur τ times within a period. As a consequence, the growth of *aggregate* consumption $C(t)$ follows the *truncated Lévy flight* explained in Sect. 2. It is a "truncated" Lévy flight because τ is finite.

We make an important assumption that there is a lower bound constraint to the elementary micro growth process. That is, the sector size after an elementary (micro) growth must be above the minimum, $c_{\min}(t)$ defined by

$$c_{\min}(t) = qc_{\mathrm{av}}(t), \qquad (0 < q < 1) \tag{12}$$

where

$$c_{\mathrm{av}}(t) = \frac{C(t)}{N}. \tag{13}$$

Here, q is the fraction of the average consumption that serves as the lower bound to all of c'. Thus, we actually obtain $c_i'(t)$ not as (8) but as

$$c_i'(t) = \max\left\{\lambda c_i(t), c_{\min}(t)\right\} = \max\left\{(1 \pm \gamma)c_i(t), qc_{\mathrm{av}}(t)\right\}. \tag{14}$$

By scaling c_i by $c_{av}(t)$ we define the fraction $y_i(t)$:

$$y_i(t) = \frac{c_i(t)}{c_{\mathrm{av}}(t)}.$$

It satisfies the normalization condition that the average of the fractions is 1:

$$\bar{y} = \frac{1}{N}\sum_i y_i(t) = 1. \tag{15}$$

By changing variables into

$$Y_i = \ln y_i, \tag{16}$$

we observe that the basic dynamics defined by (14) becomes a kind of random walk with varying step sizes, i.e. a *truncated Lévy flight* explained earlier with a lower reflecting barrier:

$$Y_i'(t) = Y_i(t) + \ln \lambda. \tag{17}$$

Here again, the prime indicates *the value after one elementary event*, *not* the *calender time* derivative.

The master equation is

$$P(Y'(t)) - P(Y(t)) = \frac{1}{N}\left[\int \mathrm{d}F(\lambda)P(Y - ln\lambda, t) - P(Y, t)\right] \tag{18}$$

where F is the probability distribution for λ.[5] Each sector is assumed to be selected with equal probability $1/N$, here. It is shown by Choquet (1960), and Choquet and Deny (1960), Lévy and Solomon (1996) that the asymptotic stationary distribution of Y, $P(Y)$ defined by

$$\int_q^N \mathrm{d}F(\mu)P(Y-\mu)\mathrm{d}\mu = \Lambda P(Y) \qquad (\Lambda \quad \text{is a constant}) \tag{19}$$

is exponential. That is, we obtain

$$P(Y) \propto \mathrm{e}^{-\alpha Y}. \tag{20}$$

Since Y is defined as $\ln y$ (Eq. 16 above), we have the probability density function of y as

$$p(y) = Ky^{-1-\alpha}. \tag{21}$$

This constant K is determined by the fact $p(y)$ is the probability density that integrates to 1:

$$\int p(y)\mathrm{d}y = 1. \tag{22}$$

At the same time, the normalization noted earlier requires that the mean of y is 1 (Eq. 15):

$$\int yp(y)\mathrm{d}y = 1 \tag{23}$$

[5] In our present model, $\lambda = 1 \pm \gamma$ (Eq. 9). Thus, λ is assumed to take only two values. More generally, if a probability density function $f(\lambda)$ exists for λ, then $\mathrm{d}F(\lambda)$ in (18) should be replaced by $f(\lambda)\mathrm{d}\lambda$.

From (22) and (23), we can derive an implicit relation between q, α and N (See Malcai et al. (1999));

$$N = \left(\frac{\alpha - 1}{\alpha}\right)\left[\frac{(q/N)^{\alpha} - 1}{(q/N)^{\alpha} - (q/N)}\right]. \tag{24}$$

This equation is solved approximately as

$$\alpha \approx \frac{1}{1 - q}, \tag{25}$$

when $N \gg \mathrm{e}^{1/q}$.

Denote by $R(r;\tau)$ the cumulative distribution function of r, that is, the probability that the growth rate is less than or equal to r with τ elementary growths. We also define

$$\bar{R}(r;\tau) = 1 - R(r;\tau). \tag{26}$$

For $\tau = 1$, we define

$$S(r) = R(r, 1) \tag{27}$$

and

$$\bar{S}(r) = 1 - S(r). \tag{28}$$

Note that the distribution function S refers to one elementary change to only one of the N sectors.

From the asymptotic solution of the master equation, we obtain

$$\bar{S}(r) \sim \left(\frac{r}{r_{\min}}\right)^{-\alpha} \tag{29}$$

where

$$r_{\min} = \frac{\gamma c_{\min}}{C} = \frac{\gamma q c_{\mathrm{av}}}{C} = \gamma q \left(\frac{C}{N}\right)\left(\frac{1}{C}\right) = \frac{\gamma q}{N}. \tag{30}$$

There are many ways how the growth rate of *aggregate* consumption, r is realized.[6] The same aggregate growth rate, r, may be due either to a small number of elementary micro growths with large step size such as, $r/2$ and $r/3$, or to a large number of small micro growths each with small step size. This makes a difference to the emerging probability distribution of r.

[6] A positive r and a negative r can be treated in almost identical ways. We focus on positive r.

3.2 Exponential distribution

In what follows, we will derive an exponential distribution for the aggregate growth rate r, under the condition that the number of micro events τ does not exceed a critical level, $\bar{\tau}$ which is defined shortly. It is simpler to explain this by way of a concrete example.

Suppose that there are $2k$ events out of τ with magnitude $\gamma/2$ rates each, and that the rest of τ (i.e. $\tau - 2k$) steps makes almost zero net contribution to the aggregate growth. In this case, the aggregate growth rate r is achieved by way of

$$r = 2k\left(\frac{\gamma}{2}\right) = k\gamma \quad (k = 1, 2, \dots). \tag{31}$$

The probability that the aggregate growth rate is equal to or greater than $k\gamma$ with τ elementary events is given by

$$\bar{R}(r = k\gamma, 2k) = \binom{\tau}{2k}\left[\bar{S}\left(\frac{\gamma}{2}\right)\right]^{2k} \cong \tau^{2k}\left[\bar{S}\left(\frac{\gamma}{2}\right)\right]^{2k}. \tag{32}$$

Here, we have used the fact that the number of combinations of taking $2k$ out of τ, that is $\tau C_{2k} = \tau(\tau-1)\dots(\tau-2k+1)/(2k)!$ can be approximated by τ^{2k}. We know from (29) and (30)

$$\left[\bar{S}\left(\frac{\gamma}{2}\right)\right]^{2} = \left(\frac{N}{2q}\right)^{-\alpha}. \tag{33}$$

Thus, we can rewrite the above probability (32) as

$$\bar{R}(r = k\gamma, 2k) = \left[\left(\frac{N}{2q}\right)^{2\alpha}\frac{1}{\tau^2}\right]^{-r/\gamma} \tag{34}$$

If we define

$$\bar{\tau} = \left(\frac{N}{2q}\right)^{\alpha}, \tag{35}$$

we can rewrite Eq. (34) more compactly as follows:

$$\bar{R}(r = k\gamma, 2k) = \left[\frac{\bar{\tau}^2}{\tau^2}\right]^{-r/\gamma}. \tag{36}$$

When $\tau < \bar{\tau}$ is satisfied, then this probability distribution is exponential. It is given by

$$\bar{R}(r = \gamma k, 2k) = \exp\left[-\left(\frac{\ln A}{\gamma}\right)r\right] \tag{37}$$

where

$$A = \left(\frac{\bar{\tau}}{\tau}\right)^2.$$

Its density function is of the form

$$f(r) = -\frac{\mathrm{d}\bar{R}(r, 2k)}{\mathrm{d}r} = \frac{\ln A}{\gamma} \exp\left[-\left(\frac{\ln A}{\gamma}\right) r\right]. \tag{38}$$

We note that the peak value of this density is

$$f(0) = \frac{\ln A}{\gamma} \tag{39}$$

The probability of the rate r being achieved as $r = bk \times \gamma/b, \quad (b = 1, 2, \ldots)$ can be calculated similarly. No substantial changes are involved.

Thus, we have established the following proposition.

Proposition 1. *To obtain an exponential distribution for the growth rate of aggregate consumption, the number of elementary events within a given calender time period, τ cannot exceed a critical level $\bar{\tau}$. This maximum number $\bar{\tau}$ is given by*

$$\bar{\tau}(b) \approx \frac{(N)^{\alpha}}{(bq)}$$

for some small positive integer b. The probability density function of r is then the following exponential distribution:

$$f(r, b) \propto \exp\left[-\frac{b}{\gamma} \log\left(\frac{\bar{\tau}}{\tau}\right) r\right]$$

□

The probability distribution of r depends on parameters. In particular, it depends crucially on τ and γ. Figure 2 shows the region where we obtain exponential distribution in terms of τ and γ. For $\tau < \bar{\tau}$, and r larger than γ, we obtain the exponential distribution of r.

Note that in our example, r is $k\gamma$ (Eq. (31), $k = 1, 2, \ldots$), and, therefore, is necessarily greater than γ. γ is the growth rate of "elementary events" (Eq. 9) so that we can conceptually take it as small as we wish.

The implication of Proposition 1 is that to obtain exponential distribution for the growth rate of *aggregate real* variable as we actually do, the number of micro growths within a short period of time, τ must be sufficiently "small."

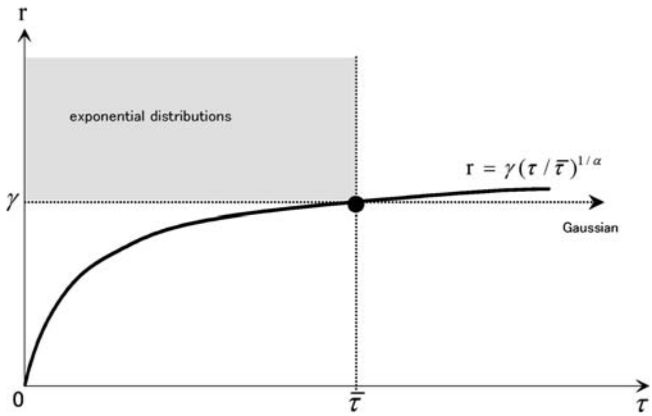

Fig. 2 Exponential distribution for the real economy (see Huang and Solomon (2001) for further details of this diagram)

3.3 Financial returns

In this section, we study a similar model of financial returns, and characterize the case where power-law distribution holds. There are N agents or assets each with financial resources or wealth, $w_i(t)$:

$$W(t) = w_1(t) + \cdots + w_N(t). \tag{40}$$

As in the real-sector model, one of N sectors (or stocks) are randomly selected for an elementary event, i.e. a micro change. This random selection could be uniform with probability $1/N$, or could be modified to favor large sectors (or investors). There are τ such micro or elementary events (that is, transactions) within a unit interval of time.

When sector i is selected, it undergoes the change

$$w_i'(t) = (1+g)w_i(t) \tag{41}$$

where

$$g = \pm\gamma.$$

Again, $w_i'(t)$ indicates the value of w_i immediately after the elementary growth in the ith asset or agent's wealth. It is *not* the time derivative of $w_i(t)$.

The rate of return on financial assets or wealth over a calender period from t to $t+1, r(t)$ is defined, analogously as the rate of growth of consumption, by

$$r(t) = \frac{W(t+1) - W(t)}{W(t)}. \tag{42}$$

We are interested in the case where probability distribution of r becomes power distribution. When there are τ elementary events during $[t, t+1]$, r is denoted by $r(t,\tau)$. By definition, $r(t,\tau)$ is as follows.

$$r(t,\tau) := \sum_{i,k} f_{i,k} \tag{43}$$

where

$$f_{i,k} = \frac{\pm\gamma w_i(t;k)}{W(t)}.$$

As in the real sector model, we obtain

$$\bar{S}(r) = 1 - R(r;1) \sim \left(\frac{r}{r_{\min}}\right)^{-\alpha} \tag{44}$$

where

$$r_{\min} = \frac{\gamma w_{\min}}{W} = \frac{\gamma q w_{\mathrm{av}}}{W} = \frac{\gamma q W}{WN} = \frac{\gamma q}{N}.$$

Equation (44) corresponds to (29) for the real economy.

Again, the probability distribution of r depends on parameters, τ and γ. Huang and Solomon (2001) demonstrate that depending on parameters, power law distribution emerges. The region in which power laws *with the exponent α close to 3* emerge[7] in the financial model is shadowed in Fig. 3. The boundaries of the region are determined by two curves on the $\tau - r$ plane.

One is

$$r = \gamma \left[\frac{\tau}{\bar{\tau}}\right]^{1/\alpha} \tag{45}$$

where

$$\bar{\tau} = \left(\frac{N}{2q}\right)^{\alpha}. \tag{46}$$

The other curve is defined by

$$r = \gamma\sqrt{\frac{\tau}{\bar{\tau}}}. \tag{47}$$

Equation (45) can be derived as follows. For a given aggregate growth rate r, we compare two cases. In case one, r is attained by two elementary growths

[7] Huang and Solomon (2001) demonstrate by their simulations that the exponent α becomes close to 3.

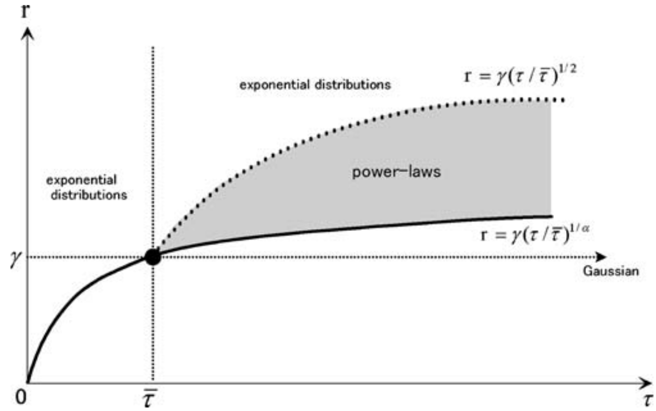

Fig. 3 Power-law distribution ($\alpha > 2$) for the financial markets

each of which has the size of $r/2$ whereas in case two, r is attained by just one elementary growth. We consider the condition that the former is more likely than the latter, that is

$$\tau C_2 \left(\bar{S}\left(\frac{r}{2}\right)\right)^2 > \tau \bar{S}(r). \tag{48}$$

Here, τC_2 is the combination of taking two elementary events with net contribution to r out of τ events, and $\bar{S}(r)$ is

$$\bar{S}(r) = \bar{R}(r, 1) = \left(\frac{r}{r_{\min}}\right)^{-\alpha} = \left(\frac{rN}{q\gamma}\right)^{-\alpha} \tag{49}$$

Recall that $\bar{S}(r)$ is the probability that at least r growth rate is attained by one elementary event. By substituting (49) into (48), and approximating $\tau C_2 \simeq \tau^2$, we can rewrite the above inequality (48) as

$$r < \bar{r} \equiv \frac{2q}{N} \gamma \tau^{1/\alpha} \tag{50}$$

In this region, power-law distributions are less likely to hold because the effects of the initial power-law for a single step, $\bar{S}(r)$ on the distribution of r weakens. Recall that the inequality (48) means that r is more likely to be obtained by two steps than a single step. In other words, power laws emerge in the region whose boundary is

$$r = \frac{2q}{N} \gamma \tau^{1/\alpha} \tag{51}$$

With $\bar{\tau}$ defined by (46), Eq. (51) can be rewritten as Eq. (45).

Likewise, we can derive the other boundary, Eq. (47); See Huang and Solomon (2001) for details. Here, we focus on the region[8] where $\tau > \bar{\tau}$ and $r > \gamma$ hold. Recall that we obtain the exponential distribution when $\tau < \bar{\tau}$ and $r > \gamma$ hold(for the real economy).

As for power - laws, we can also derive them by way of using Langevin equation. This alternative approach is explained in Appendix 1. The merit of the Huang–Solomon model is that it nests both power-laws and exponential distributions. We can summarize the results of our analysis as follows.

Proposition 2. *In a truncated Lévy flight model in which the aggregate growth rate, r is composed of a number of micro or elementary growths within a unit interval of time, the probability distribution of r depends crucially on the number of such micro events, τ. Specifically, when τ is smaller than a critical value $\bar{\tau}$, the exponential distribution emerges while we can obtain power laws with the exponent α close to 3 for $\tau > \bar{\tau}$. To the extent that the number of micro growths within a period is small in the real economy whereas it is large in financial markets, we can explain the stylized fact that we observe the exponential distribution for "real" growth whereas power laws with the exponent α close to 3 for financial returns.* □

4 Concluding remarks on real and financial markets

People often think or perhaps feel that real and financial markets are different. It rarely happens that our salaries and wages are doubled within a relatively short period of time, say a year. In contrast, we know that if not very often, the price of a stock can double in a year. This difference is formally reflected in two different probability distributions; one is exponential distribution for the growth rate of *real* variable such as GDP or consumption, while the other is power-law distribution for asset prices and returns.

We began with the observation that a particular type of multiplicative process, *the truncated Lévy flight* nests a broad range of growth processes including real and financial activities. The preceding analysis has shown that within this framework, the crucial parameter is the number of elementary micro growth events within a given period, say a month (τ in the model). When the number of micro growth events within a period is small, exponential distribution can emerge. Conversely, when the number of micro growth events within a period is large, then power-law distributions can emerge.

Thus, given the present model, to account for the stylized fact that we have exponential distribution for the growth rate of real GDP whereas power-law distributions for asset prices and returns, we must assume that within a given period, the number of micro growth events is relatively small in the case of real economic activities whereas it is large in the case of asset prices. Here, we

[8] We note that as τ becomes much larger than $\bar{\tau}$, the probability distribution eventually approaches the Gaussian distribution via intermediate Lévy like distributions. See Huang and Solomon (2001).

must take Proposition 1 as an *assumption*, and leave it for further research. We maintain that this is a plausible assumption, though. Note that what matters is not *additive* disturbance but multiplicative disturbance. We believe that we can reasonably argue that the frequency of multiplicative shocks is much higher for asset prices than for real micro economic activities.

Think of it this way. In your "real" life, the costs of transportation and food today is more or less the same as yesterday, and will be about the same tomorrow. In contrast, in stock market, you must be always aware of the presence of "multiplicative shocks." Our analysis shows that the frequency of multiplicative shocks is a crucial determinant of different probability distributions of aggregate growth rate; One exponential distribution for the real economy, and the other power laws for financial returns. In any case, it is an important research agenda to ascertain this Proposition 2. For the moment, we conclude that we have a good deal of empirical observations to reject the standard asset price model based on the representative consumer, and at the same time, a plausible theoretical reason to believe that the real economy and asset markets are different creatures.

Appendix 1: power laws derived from the Langevin equation

In this appendix, we sketch an alternative approach based on *the Langevin–equation*. This approach is known to be equivalent to the Fokker–Planck equation, and describes the stationary distributions of returns. This approach is not so helpful to our underlying the mechanisms generating financial returns and real economic activities as the one described in the main text, but is theoretically clear cut. We follow Richmond and Solomon (2001).

Let $\phi(t)$ be the instantaneous return of some financial asset. We assume that its dynamics is given by the Langevin equation:

$$\frac{\mathrm{d}\phi(t)}{\mathrm{d}t} = F(\phi(t)) + G(\phi(t))\eta(t),$$

where η is a mean zero and finite variance noise:

$$E(\eta(t)) = 0, \quad \text{and} \quad E[\eta(t)\eta(t')] = 2D\delta(t - t').$$

Richmond and Soloman (2001) have shown that the probability distribution of ϕ satisfies the generalized Fokker–Planck equation[9]

$$\frac{\partial P}{\partial t} = D\frac{\partial}{\partial \phi}\left[G\left(\frac{\partial(GP)}{\partial \phi}\right)\right] - \frac{\partial}{\partial \phi}(FP).$$

[9] See Aoki (2002, Sect. 8.7) or Honerkamp (1998, chapter 6), for example

He shows that this equation has the steady-state solution

$$P(\phi) = \frac{1}{Z|G(\phi)|} \exp(-\Psi(\phi))$$

where Z is the normalization constant, and

$$\Psi(\phi) = -\frac{1}{D} \int^{\phi} \frac{F}{G^2} \, \mathrm{d}\phi.$$

For a simple example, suppose that

$$G(\phi) = \phi + \epsilon,$$

and

$$F(\phi) = -J\phi,$$

with $J > 0$. Then, by taking $\epsilon \longrightarrow 0$, we obtain

$$P(\phi) = \frac{1}{Z} |\phi|^{-1-J/D}.$$

With a cubic

$$F(\phi) = -J\phi + b\phi^2 - c\phi^3,$$

there is a correction term $\exp((2b\phi - c\phi^2)/D$. Thus in regions where the exponential correction is nearly 1, we have an approximate power law.

Next, partially following Richmond and Solomon (2001), we apply this basic tools in modeling financial model with N agents with resources $w_i(t)$, $i = 1, 2, \ldots, N$.

The aggregate asset is

$$W(t) = \sum_j b_j w_j$$

where

$$\sum b_j = 1.$$

We focus on the fraction

$$x_i(t) = \frac{w_j(t)}{W(t)}.$$

Its Langevin equation is

$$\begin{aligned} \mathrm{d}x_i(t) &= \frac{\mathrm{d}w_i(t)}{W(t)} - \frac{w_i(t)\mathrm{d}u}{W^2(t)} \\ &= (\epsilon_i(t)\sigma_i - a)x_i(t) + a_i \end{aligned}$$

where

$$a = \sum_i b_i a_i(t)$$

We assume that the evolution of financial asset of type i agents is given by

$$\mathrm{d}w_i(t) = \epsilon_i(t)\sigma_i w_i(t) + a_i W(t), \quad i = 1, 2, \ldots, N.$$

We take without loss of generality that the mean of ϵ is 0 and variance 1.
The Langevin equation becomes

$$\mathrm{d}x_i(t) = \epsilon\sigma_i x_i(t) - a x_i(t).$$

From the formula developed above of Richmond and Solomon (2001), the steady-state solution is

$$P(x_i) = \frac{1}{(\sigma_i x_i)^2} \exp\left(2\int \frac{[-a x_i + a_i]}{(\sigma_i x_i)^2}\mathrm{d}x_i\right).$$

In the range where $2a_i/x_i\sigma_i^2 << 1$, this distribution is approximately power law:
In other words, we have

$$P(x_i) \approx x^{-1-\alpha_i},$$

where

$$\alpha_i = 1 + \frac{2a}{\sigma_i^2.}$$

Appendix 2: examples of power laws in clusters

In the main text, we have considered a truncated Lévy flight model in which power law distribution is generated. In that model, lower bounds or lower reflective boundaries play the essential role for generating power laws. There are different models. This appendix assembles such models in which power laws are generated without the assumption of lower bounds.

1. Large deviations and power laws

For i.i.d. Poisson random variables X_i, $i = 1, 2, \ldots$, the sum possesses a power-law relation. Let $S_n = X_1 + \cdots + X_n$. By the Markov or Chernoff inequality

$$\Pr \frac{(S_n)}{(n > a)} = \Pr(e^{\theta S_n} \geq e^{na\theta}) := e^{-nI(a)},$$

where θ is non-negative, and

$$I(a) := \sup[\theta a - lnM(\theta)],$$

where $M(\theta) = E(e^{\theta X_1})$.

When x is Poisson distributed with mean λ, then $M(\theta) = \exp[\lambda(e^{\theta} - 1)]$, and $I(a) = a \ln(a/\lambda) - (a - \lambda)$. We have a power- law relationship

$$\Pr(S_n/n > a) \approx a^{-na}.$$

□

2. Yule process

In a Yule process a cluster stars with a single agent. New agents arrive as a linear birth process with rate λ. These agents are all of the same type. Let $N(t)$ be the number of agents at time t. Its probability generating function is

$$G(z,t) = \sum_{i=0}^{\infty} p_i z^i = \left[\frac{ze^{-\lambda t}}{(1 - z + ze^{-\lambda t})}\right]^{n_0},$$

where p_i is the probability that $N(t) = i$, and n_0 is the initial number of agents, which is set to 1. See Cox and Miller (1965) for example. From this generating function we retrieve

$$p_n(t) = e^{-\lambda t}(1 - e^{-\lambda t})^{n-1}.$$

Aldous (2001) has an example which modifies this basic Yule process model by assuming that new types of agents appear within each cluster at constant rate μ, so that the number of types grow exponentially. The number of agents in a randomly chosen cluster is given by

$$p(n) = \int_0^{\infty} \mu e^{-\mu t} e^{-\lambda t}(1 - e^{-\lambda t})^{n-1} dt.$$

Changing variable from t to $x = \mathrm{e}^{-\lambda t}$, we recognize this integral to be that of a Beta integral, hence

$$p(n) = \frac{\Gamma(1+\rho^{-1})}{\rho} \frac{\Gamma(n)}{\Gamma(n+1+\rho^{-1})} \approx n_1 - \rho^{-1},$$

with $\rho = \lambda/\mu$ □.

3. Birth–death process with second-order balanced rates

A model related to this arises in the birth–death process when their rates are

$$\frac{\lambda_{i-1}}{\mu_i} = 1 - \frac{a}{i} + O\left(\frac{1}{i^2}\right) = \left[1 + \frac{a}{i}\right]^{-1} + O\left(\frac{1}{i^2}\right),$$

where λ_{i-1} is the birth rate in a cluster of size $i-1$, and μ_i is the death rate of a cluster of size i. This assumption says that clusters of small sizes have higher death rates than larger clusters.

Let μ be the entry rate of a singleton. We can write down the master equation for this model and look for stationary solutions by means of the detailed balance condition. We also assume that the maximum size is N.

The last of the master equation is then

$$\mathrm{d}p_N(t)/\mathrm{d}t = \lambda_{N-1} p_{N-1}(t) - \mu_N p_N(t).$$

The total number of clusters is given by

$$F_N = \sum_{i=1}^{N} p_i(t)$$

and the fraction of clusters of size i is

$$a_i(N) = \frac{p_i}{F_N},$$

where p_i is the stationary state value of $p_i(t)$ as t goes to infinity.

It is straightforward to verify that

$$a_i(N) \prod_{j=2}^{i} \left(1 + \frac{a}{j}\right)^{-1} \approx i^{-a}.$$

This is the power - law for this model. See Karev et al. (2002). □

4. Yule–Simon model

Ewens distribution yields Zifp distribution and not more general power law, since the expected number of clusters of size i in the presence of n agents in the model, $E(a_i(n)$ is given by

$$z_i(n) := E(a_i(n)) = \frac{\theta}{i}\left(1 - \frac{i}{n}\right)^{\theta-1} \approx \frac{\theta}{i}.$$

See Aoki (2002, p. 156). The Ewens model has the entry rate of new type $\theta/[\theta + n]$ which depends on n.

Costantini et al. (2005) has introduced entry rate which is independent of n in their reformulation of Yule (1924) and Simon (1955).

Let u be the probability of starting a new cluster by an entrant. Then, $1 - u$ is the probability of a new entrant joining one of the existing clusters.

Denoting the expected value of $a_i(n)$ by $z_i(n)$ as above, we have, recalling that an entry into a cluster with n changes the number of clusters of i and $i - 1$

$$z_i(n+1) - z_i(n) = (1-u)\left(\frac{[(i-1)z_{i-1}(n) - iz_i(n)]}{n}\right),$$

and

$$z_i(n+1) - z_i(n) = u - \frac{(1-u)z_1(n)}{n}.$$

We focus on a stationary relation of this recursive relations

$$\frac{z_i^*(n+1)}{n+1} = \frac{z_i^*(n)}{n}.$$

We derive the relation

$$z_i^*(n) = \frac{i-1}{\rho_i} z_{i-1}^*(n),$$

where $\rho = 1/(1-u)$.

The stationary solution has the Yule distribution

$$f_i = \frac{z_i^*(n)}{nu} = \rho B(i, \rho + 1), \quad i = 1, 2, \ldots.$$

The fraction sums to one and we have

$$f_i \approx i^{-1-\rho}.$$

□

5. The Chinese restaurant process and its variants

Dubins and Pitman proposed a construction for cluster formation which is known as the Chinese restaurant process. A restaurant has an infinite number of tables and each table can seat an infinite number of customers (agents). The first agent sits at a table. Call it table 1, or cluster 1. After n agents have been seated, there are k_n occupied tables, with $n = n_1 + \cdots + n_{k_n}$. This is the random partition of n into k_n subsets. The next customer either sits at one of the already occupied tables with equal probability, or starts a new table.

There is a one parameter version and a two-parameter version (due to Pitman (2002)). In the two-parameter version the probability of starting a new table is $(\theta + k_n\alpha)/(\theta + n)$, and the probability of sitting at a table with n_i guests already seated is $(n_i - \alpha)/(n + \theta)$. In the one-parameter version $\alpha = 0$, and θ is some positive number.

Agents i and j are of the same type, that is, they belong to the same partitioned subset of $[n] := \{1, 2, \ldots, n\}$, if seated at the same table.

Pitman (2002) has shown that the sequence $x_1, x_2, \ldots$ defined as the fraction of the partitioned subsets divided by n converges to the Poisson–Dirichlet distribution as n goes to infinity, denoted by $\mathrm{PD}(\alpha, \theta)$. The kth term of the sequence is given by $D_k \prod_{j=1}^{k-1}(1 - D_j)$, where D_j is distributed as $B(1 - \alpha, \theta + j\alpha)$.

Durrett and Schweinsberg (2005) modified this process by assuming that the $(n + 1)$th cluster (table) is started with a constant probability r.

More generally, they prove that the cluster size distribution has the power laws. □

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