ON TANGENTIAL APPROACH REGIONS FOR BOUNDED HARMONIC FUNCTIONS IN THE UNIT DISC

FAUSTO DI BIASE, ALEXANDER STOKOLOS, OLOF SVENSSON, AND TOMASZ WEISS

ABSTRACT. We study bounded harmonic functions defined on the unit disc and their boundary behaviour along tangential approach regions whose shape may change from point to point, thus solving a problem posed by W. Rudin in 1979 and completing the picture given by the basic theorems of Fatou (1906), Littlewood (1927) and Nagel & Stein (1984).

1. MOTIVATION AND RESULTS

We study the boundary behaviour of bounded harmonic functions, defined in the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, along tangential approach regions whose shape may possibly change from point to point. The boundary of $\mathbb{D}$, denoted $\partial\,\mathbb{D}$, is the set $\{w \in \mathbb{C} : |w| = 1\}$; it is identified, via the map $s \in \mathbb{R} \mapsto e^{is} \in \partial\,\mathbb{D}$, to the quotient group $\mathbb{R}/2\pi\,\mathbb{Z}$, from which it inherits the *Lebesgue measure* m; thus, $m(\partial\,\mathbb{D}) = 2\pi$.
The first motivation of the present work comes from our desire to completely clarify a claim made in the introduction to Littlewood (1927), where the Author proved the failure of almost everywhere convergence of bounded holomorphic functions in the unit disc along *rotated* copies of any given curve in the unit disc ending *tangentially* to the boundary. This 'negative' theorem of Littlewood complements the 'positive' theorem of Fatou (1906), where the almost everywhere *nontangential* convergence of bounded holomorphic functions in the unit disc is established. In his paper, Littlewood claims that there is

> no possible question, in the negative theorem, of allowing [the tangential curve] to vary its shape [from point to point].

We provide the needed clarification of this claim and articulate our results[1] in four theorems, only the first of which may be fairly considered to be within the reach of 1927 technology or amenable to Littlewood's proof.
The second motivation, closely connected to the first, comes from our desire to give a complete answer to a question asked in Rudin (1979), where the Author looks at Littlewood's claim from the positive side and asks whether almost everywhere convergence of bounded holomorphic functions in the unit disc could possibly hold

Date: October 14, 2003.
2000 *Mathematics Subject Classification.* Primary 31A20; Secondary 03E15.
Key words and phrases. Harmonic functions, inner functions, boundary behaviour, almost everywhere convergence, Lebesgue points, tangential curves, tresses, independence proofs, unit disc.
Thanks to the Harmonic Analysis Network of the European Commission, the Wenner-Gren Foundation, Istituto Nazionale di Alta Matematica, Consiglio Nazionale delle Ricerche, and to the Universities of Luleå, Roma 'Tor Vergata' and Chieti-Pescara and the Chalmers Institute of Technology for financial support and/or hospitality.

[1] A preliminary version of some of our results is announced in Di Biase *et al.* (1998)

along a given family of *tangential* curves, ending at the various boundary points — by Littlewood's theorem, this hypothetical 'good' family of tangential curves could *not* possibly be rotation invariant.
The crucial property about curves is isolated by the following notion.

Definition. A *tress* is a family $\gamma = \{\gamma(w)\}_{w \in \partial \mathbb{D}}$ where $\gamma(w)$ is a nonempty subset of the unit disc such that the set $\{w\} \cup \gamma(w)$ is connected, for each $w \in \partial \mathbb{D}$.

Note. We show that this notion is indeed sufficient in order to state and prove our results in a simple and natural fashion. In the notion of tress, it is important to require that the set $\{w\} \cup \gamma(w)$ is connected, in order to obtain the extension of Littlewood's theorem as in Theorem 1.4. Indeed, if the set $\{w\} \cup \gamma(w)$ is not connected, then it may consist of a Nagel–Stein type tangential sequence, for which almost everywhere convergence does indeed hold; cf. Nagel & Stein (1984). The Nagel–Stein phenomenon holds in great generality; see Di Biase (1998).

A preliminary reduction. Since the core of the problem belongs to harmonic analysis, we may restrict ourselves, without loss of generality, to the space $\mathrm{h}^\infty(\mathbb{D})$, consisting of real valued functions harmonic and bounded on $\mathbb{D}$. Indeed, Fatou (1906) also proved that for every $h \in \mathrm{h}^\infty(\mathbb{D})$, there is a measurable subset $F(h) \subset \partial \mathbb{D}$ of Lebesgue measure 2π such that for each $w \in F(h)$ the limit of $h(z)$, as $z \to w$ and $\frac{1-|z|}{|w-z|} > \delta$, exists for each $\delta > 0$; this limit is denoted $h_\flat(w)$. Thus, $h_\flat$ is an almost everywhere defined function on $\partial \mathbb{D}$ and $h_\flat \in \mathrm{L}^\infty(\partial \mathbb{D})$. Points in $F(h)$ are called *Fatou points of h.* The *Poisson integral operator* $P : \mathrm{L}^\infty(\partial \mathbb{D}) \to \mathrm{h}^\infty(\mathbb{D})$ recaptures h from $h_\flat$, since $h = P[h_\flat]$; see Fatou (1906).

Definitions. Let γ be a tress. If

$$\gamma(e^{is}w) = \{e^{is}z : z \in \gamma(w)\}$$

for each $w \in \partial \mathbb{D}$ and $s \in \mathbb{R}$, then γ is called *rotation invariant.* If each set $\gamma(w)$ is *tangential to $\partial \mathbb{D}$ at w*, i.e., for each $w \in \partial \mathbb{D}$ and $\epsilon > 0$, there is $\delta > 0$ such that if $z \in \gamma(w) \cap \mathbb{D}$ and $|z - w| < \delta$ then $\frac{1-|z|}{|z-w|} < \epsilon$, then γ is called *tangential.* If, for each $w \in \partial \mathbb{D}$, there is a continuous map $c_w : [0, \infty) \to \mathbb{D}$ whose image is equal to $\gamma(w)$ and such that $\lim_{\tau \to \infty} c_w(\tau) = w$ then γ is called a *tress of curves.*
Let $h \in \mathrm{h}^\infty(\mathbb{D})$. The *convergence set of h along γ*, denoted $\mathrm{C}(h, \gamma)$, is the set consisting of those points $w \in F(h)$ such that h converges to $h_\flat(w)$ along $\gamma(w)$, i.e., for each $\epsilon > 0$ there is $\delta > 0$ such that if $z \in \gamma(w)$ and $|z - w| < \delta$ then $|h_\flat(w) - h(z)| < \epsilon$. The *divergence set of h along γ*, denoted $\mathrm{D}(h, \gamma)$, is the set consisting of those points $w \in \partial \mathbb{D}$ such that $h(z)$ does not converge to any number as $z \to w$ and $z \in \gamma(w)$.

Note. The divergence set and the convergent set are disjoint. However, the former is not defined to be the complement of the latter. Therefore, our results turn out to be the most stringent possible ones.

Littlewood's theorem can be restated as follows: If γ is a tangential, rotation invariant tress of curves then there is $h \in \mathrm{h}^\infty(\mathbb{D})$ such that $m(\mathrm{D}(h, \gamma)) = 2\pi$. Our Theorem 1.4 shows that, Littlewood's claim notwithstanding, it is possible, in the 'negative theorem', to allow the approach regions to change their shape from point to point, i.e., it is indeed possible to prove a Littlewood type theorem for tangential tresses that are not assumed to be rotation invariant, the only extra hypothesis being a natural condition, of a qualitative nature, to be given below. However, Littlewood's claim can be given the following precise rendition.

Theorem 1.1. *There exists a tangential tress of curves γ such that for each $h \in \mathrm{h}^\infty(\mathbb{D})$, the outer measure of the set $\mathrm{C}(h,\gamma)$ is 2π.*

Notes. Theorem 1.1 gives a precise rendition of Littlewood's claim but it also raises further questions, whose answers indicate that the claim itself, in its vague but peremptory form, did not describe the complete picture, since our answers were not accessible to 1927 technology. Moreover, in Theorem 1.4 we will show that, Littlewood's claim notwithstanding, under certain natural conditions, of a qualitative nature, it is possible to extend Littlewood's theorem to tangential tresses that are *not* assumed to be rotation invariant.

Observe that Theorem 1.1 does not guarantee that the set of convergence thereby considered is measurable, but only that its outer measure is equal to 2π. Indeed, Rudin's question may be formulated as one about *the truth value of the following statement*:

> There is a tangential tress of curves γ such that for each $h \in \mathrm{h}^\infty(\mathbb{D})$ the set $\mathrm{C}(h,\gamma)$ is measurable and it has measure 2π.

One may be tempted to apply the law of the excluded middle and deduce that the statement we have highlighted must be either true or false. However, Gödel's work warns us of other possibilities.

In order to prove a statement in Analysis, we ultimately deduce it from the axioms of Zermelo Fraenkel together with the Axiom of Choice; following the literature[2], we denote these axioms by ZFC. Gödel showed that a statement can be deduced from ZFC if and only if it holds in every model of ZFC. For example, the Continuum Hypothesis holds in some but not all models of ZFC, by results of P. Cohen (1963–1964); cf. Cohen (1966) and Kunen (1980).

A priori, it may be the case that bounded harmonic functions behave differently in different models of ZFC, but not in radically different ways. In the following Theorems 1.2 and 1.3 we prove that the boundary behaviour of $\mathrm{h}^\infty(\mathbb{D})$ functions along tresses is *radically* different in different models of ZFC.

Theorem 1.2. *There is a model of ZFC where the following statement holds: there exists a tangential tress of curves γ such that for each $h \in \mathrm{h}^\infty(\mathbb{D})$, the set $\mathrm{C}(h,\gamma)$ is measurable and it has measure 2π.*

Theorem 1.3. *There is a model of ZFC where the following statement holds: for every tangential tress γ there exists $h \in \mathrm{h}^\infty(\mathbb{D})$ such that the set $\mathrm{D}(h,\gamma)$ has outer measure equal to 2π.*

Note. It is not possible to prove, in Theorem 1.3, that the set $D(h,\gamma)$ can be made measurable and of measure 2π, because Theorem 1.1 is a theorem in ZFC and therefore it holds in every model of ZFC.

Littlewood's claim notwithstanding, it is indeed possible to extend Littlewood's theorem to families of tangential approach regions that are *not* assumed to be rotation invariant, the only extra hypothesis being given in the following natural (but novel) condition, of a qualitative nature.

[2]For background on these matters, see Drake (1974), Jech (1978).

Definition. A tress γ is *regular* if for each open set $O \subset \mathbb{D}$, the subset of $\partial\,\mathbb{D}$ given by

$$\gamma^{\downarrow}(O) \overset{\text{def}}{=} \{w \in \partial\,\mathbb{D} : O \cap \gamma(w) \neq \emptyset\}$$

is a measurable subset of $\partial\,\mathbb{D}$.
The set $\gamma^{\downarrow}(O)$ is called the *shadow projected by O along γ*.

Examples. A rotation invariant tress is necessarily regular, since its shadows are *open* subsets of $\partial\,\mathbb{D}$. Other interesting and equally natural examples of regular tresses are given by the inner function images of radii. An *inner function* is an analytic function $f : \mathbb{D} \to \mathbb{D}$ whose nontangential limit $f_\flat(w)$ belongs to $\partial\,\mathbb{D}$ for almost every $w \in \partial\,\mathbb{D}$. If f is an inner function then almost every point $u \in \partial\,\mathbb{D}$ is equal to $f_\flat(w)$ for at least one $w \in \partial\,\mathbb{D}$. If $u \in \partial\,\mathbb{D}$ and if there is at least one point $w \in \partial\,\mathbb{D}$ such that $f_\flat(w) = u$ then we define $f_\star(u) \overset{\text{def}}{=} \{f(rw) : 0 \leq r < 1, f_\flat(w) = u\}$; the definition of $f_\star$ at the other points of $\partial\,\mathbb{D}$ is not influential since those points form a null set; then $f_\star$ is a regular tress.

Note. The set $\gamma^{\downarrow}(B)$, where B is the boundary of a *sawtooth regions*, appear (implicitly) in the proof by A. Calderón of the so-called *local Fatou theorem*[3]. We now sketch the technique, due to E.M. Stein, showing the relevance of the sets $\gamma^{\downarrow}(O)$ in the study of the nontangential maximal functions, i.e. when $\gamma = \Gamma$ is the family of nontangential approach regions of fixed width, in the upper half space $\mathbb{R}^n \times (0, \infty)$ and O is an open subset of the upper half space (obtained as superlevel set of a certain function). In this case, it follows that the shadow $\Gamma^{\downarrow}(O)$ is *open* and, therefore, one may apply a Whitney type decomposition in order to control the relevant nontangential maximal function; see Fefferman & Stein (1971) and Stein (1993). For extensions and other applications, see Di Biase (1998).

Theorem 1.4. *For each regular, tangential tress γ, there exists $h \in \mathrm{h}^\infty(\mathbb{D})$ such that the set $\mathrm{D}(h, \gamma)$ is measurable and has measure 2π.*

Note 1. Our notions of *tress* and *regular tress* appear to be the weakest qualitative properties forcing a family of tangential approach regions to yield to the conclusion of a Littlewood's type theorem, such as the one we obtain in Theorem 1.4. The very statement of Theorem 1.4 is a significant extension of Littlewood's theorem. The proof of Theorem 1.4 is achieved using techniques quite different than those used by Littlewood, whose proof was based on

> a result of Khintchine concerning the rapidity of the approximation
> of almost all numbers by rationals.

The quote is from Zygmund (1949) where a simpler proof of Littlewood's theorem, avoiding the use of Khintchine's theorem, is given. Our proof of Theorem 1.4 is based on Zygmund's technique.

Note 2. Theorem 1.4 recaptures some results of Rudin (1988), proved using complex analysis methods and under more stringent hypothesis. Our use of purely real variable methods makes it possible to extend our results to higher dimensional situations[4].

---

[3]The local Fatou theorem for harmonic functions is a real-variable higher dimensional version of Privalov's extension of Fatou's theorem; see Stein & Weiss (1971) and references therein.

[4]We intend to elaborate these extensions in a forthcoming paper. See also Aikawa (1990) and Aikawa (1991).

2. Proof of Theorem 1.1

We shall need the existence of a special partition of $\partial\mathbb{D}$ in a continuous family of disjoint sets of full outer measure.

Lemma 2.1 (Lusin & Sierpiński (1917)). *There is a collection $\{G_u\}_{u\in(0,1)}$ of mutually disjoint subsets of $\partial\mathbb{D}$, such that (a) for each $u \in (0,1)$, the set G_u has outer measure equal to 2π; (b) $\partial\mathbb{D} = \bigcup_{u\in(0,1)} G_u$.*

Now, we recall the following qualitative[5] consequence of Fatou's theorem.

Lemma 2.2. *For each $h \in h^\infty(\mathbb{D})$ there exists a tangential tress of curves γ_h such that the set $\mathrm{C}(h,\gamma_h)$ is equal to $F(h)$ and, therefore, $m(\mathrm{C}(h,\gamma_h)) = 2\pi$.*

Proof. Let $w \in F(h)$. For each $n \in \mathbb{N}$, there is $r(n) > 0$ such that if $z \in \mathbb{D}$, $|z-w| \le r(n)$ and $\frac{|z-w|}{1-|z|} = 2^n$ then $|h(z) - h_\flat(w)| < \frac{1}{n}$; we may assume that $r(n+1) < \frac{1}{2}r(n)$ and $r(1) < 1/2$. Choose $z(1) \in \mathbb{D}$ such that $|z(1) - w| = r(1)$ and $\frac{|z(1)-w|}{1-|z(1)|} = 2$. Let $z(2)$ be the point $z \in \mathbb{D}$ located on the same side as $z(1)$ with respect to the radius ending at w and such that $\frac{|z-w|}{1-|z|} = 2$ and $|z-w| = r(2)$. Connect the points $z(1)$ and $z(2)$ with the segment of the curve $\frac{|z-w|}{1-|z|} = 2$ between them. Let $z(3)$ be the point $z \in \mathbb{D}$ located on the same side as $z(2)$ with respect to the radius ending at w such that $|z-w| = |w-z(2)|$ and $\frac{|z-w|}{1-|z|} = 2^2$. Connect $z(2)$ with $z(3)$ with the arc of the circle $|z-w| = |w-z(2)|$ between them. Proceed by induction, obtaining a curve ending tangentially at w, along which h converges to $h_\flat(w)$. □

We are now ready to prove Theorem 1.1.

Proof. Since the sets $(0,1)$ and $h^\infty(\mathbb{D})$ have the same cardinality, the result given in Lemma 2.1, yielding a decomposition of $\partial\mathbb{D}$ into sets of full outer measure, yields one such decomposition where the index set is $h^\infty(\mathbb{D})$. Thus, we have a disjoint union $\partial\mathbb{D} = \bigcup_{h\in h^\infty(\mathbb{D})} G(h)$, where each set $G(h)$ has full outer measure and sets indexed by different functions are disjoint. For $w \in G(h) \cap F(h)$ define $\gamma(w) \stackrel{\text{def}}{=} \gamma_h(w)$. For $w \in G(h) \setminus F(h)$ define $\gamma(w)$ as the set of points $z \in \mathbb{D}$ such that $1 - |z| = |z-w|^3$ and z is located on one side of the radius ending at w. Thus, γ is a tangential tress of curves. We claim that for each $h \in \mathrm{h}^\infty(\mathbb{D})$ the set $\mathrm{C}(h,\gamma)$ has outer measure equal to 2π. Indeed, it suffices to show that $\mathrm{C}(h,\gamma)$ contains $G(h) \cap F(h)$, since the intersection of a subset of $\partial\mathbb{D}$ of outer measure 2π with a measurable subset of $\partial\mathbb{D}$ of measure 2π is a subset of $\partial\mathbb{D}$ of outer measure 2π. Now, the inclusion $G(h) \cap F(h) \subset \mathrm{C}(h,\gamma)$ follows from the construction of γ. □

3. Proof of Theorem 1.2

3.1. **Analytic Preliminaries.** The results of this section are analytic results valid in any model of ZFC.
First, we need to clarify the relation between the boundary behaviour of $h \in \mathrm{h}^\infty(\mathbb{D})$ at a point $e^{is} \in \partial\mathbb{D}$ and the differentiability properties of $h_\flat$ at e^{is}.

[5] The quantitative version given in Boehme & Weiss (1971) will be useful later on.

Definition. If $h \in \mathrm{h}^\infty(\mathbb{D})$, $s \in \mathbb{R}$, $\theta > 0$ and $\mathrm{v} \in \mathbb{R}$ we define

$$h^*(s,\theta;\mathrm{v}) \stackrel{\text{def}}{=} \sup_{0<|t|\le\theta} \left| \frac{1}{t} \int_s^{s+t} (h_\flat(e^{iu}) - \mathrm{v})\, du \right| .$$

Note. The limit of $\frac{1}{t}\int_s^{s+t} h_\flat(e^{iu})\, du$ as $t \to 0$ exists and is equal to v if and only if

$$\lim_{\theta\downarrow 0} h^*(s,\theta;\mathrm{v}) = 0 .$$

Observe that $h^*(s,\theta;\mathrm{v})$ is an increasing function of θ.

Theorem 3.1 (Fatou (1906) and Loomis (1943))**.** *Let $h \in \mathrm{h}^\infty(\mathbb{D})$ and $s \in \mathbb{R}$. Then the following conditions are equivalent.*

(i) $e^{is} \in F(h)$ *and* $h_\flat(e^{is}) = \mathrm{v}$*;*
(ii) $\lim_{\theta\downarrow 0} h^*(s,\theta;\mathrm{v}) = 0$

Definition. Let c be a continuous function $c : [0,\infty) \to \mathbb{D}$ ending at e^{is} and assume that c can written in the form

$$c(\tau) = |c(\tau)| e^{is} e^{\theta(\tau)}$$

where $\theta = \theta(\tau) > 0$ is a continuous functions of τ such that $\lim_{\tau\to\infty}\theta(\tau) = 0$ and

$$\lim_{\tau\to\infty} \frac{\theta(\tau)}{1-|c(\tau)|} = +\infty .$$

Then c is called an *upper tangential curve ending at* e^{is}. The function $\theta = \theta(\tau)$ (uniquely determined by c) is called the *angle of* c with respect to e^{is}.

Theorem 3.2 (Boehme & Weiss (1971))**.** *Let $h \in \mathrm{h}^\infty(\mathbb{D})$, $e^{is} \in F(h)$ and $\mathrm{v} = h_\flat(e^{is})$. Let c be an upper tangential curve ending at e^{is} and let θ be the angle of c with respect to e^{is}. If*

$$\lim_{\tau\to\infty} \frac{\theta(\tau)}{1-|c(\tau)|} h^*(s, 2\theta(\tau);\mathrm{v}) = 0 \tag{3.1}$$

then

$$\lim_{\tau\to\infty} h(c(\tau)) = h_\flat(e^{is}) .$$

Note. Thus, h converges to $h_\flat(e^{is})$ along the tangential curve c as long as c is not too tangential, in the sense that (3.1) holds.
A diagonal type argument yields the following result.

Corollary 3.3. *If $\{h_\ell\}_\ell$ is a countable collection of elements of $\mathrm{h}^\infty(\mathbb{D})$ and $w \in F(h_\ell)$ for each $\ell \in \mathbb{N}$ then there is an upper tangential curve $c = c(\tau)$ ending at w such that $\lim_{\tau\to\infty} h_\ell(c(\tau)) = (h_\ell)_\flat(w)$ for each $\ell \in \mathbb{N}$.*

Proof. Let $\mathrm{v}_\ell \stackrel{\text{def}}{=} (h_\ell)_\flat(w)$ and write $w = e^{is}$. Define $\theta(\tau) = e^{-\tau}$. Write $H(\ell,\tau) = {h_\ell}^*(s, 2\theta(\tau);\mathrm{v}_\ell)$. Choose $k_1 > 0$ such that if $\tau \ge k_1$ then $H(1,\tau) < \frac{1}{2^{1+1}}$. Choose $k_2 > k_1$ such that if $\tau \ge k_2$ then $H(1,\tau) < \frac{1}{2^{2+2}}$ and $H(2,\tau) < \frac{1}{2^{2+2}}$. Choose $k_3 > k_2$ such that if $\tau \ge k_3$ then $H(1,\tau) < \frac{1}{2^{3+3}}$, $H(2,\tau) < \frac{1}{2^{3+3}}$ and $H(3,\tau) < \frac{1}{2^{3+3}}$. Continue inductively in a similar way. Define c so that $\frac{\theta(\tau)}{1-|c(\tau)|}$ interpolates linearly between 2^j and 2^{j+1} when τ goes from $\tau = k_j$ to $\tau = k_{j+1}$. Then $\lim_{\tau\to\infty} \frac{\theta(\tau)}{1-|c(\tau)|} H(\ell,\tau) = 0$ for each ℓ and Theorem 3.2 yields the desired result. □

3.2. **The statement of Theorem 1.2 holds in any model of ZFC in which the Continuum Hypothesis holds.**

Proof. Let I be a set having the cardinality of the continuum and let $\prec$ be a well-ordering of I (use the Axiom of Choice). It follows that if $a \in I$ then the *initial segment* $\{k \in I : k \prec a\}$ is at most countable, since we have placed ourselves in a model of ZFC in which the Continuum Hypothesis holds.
Observe that, in any model of ZFC, the set $\mathrm{h}^\infty(\mathbb{D})$ has the cardinality of the continuum, , i.e. the same cardinality as $\partial\,\mathbb{D}$.
Let $\{h_\alpha\}_{\alpha\in I}$ be a list of all bounded harmonic functions in $\mathbb{D}$ and let $\{w_\beta\}_{\beta\in I}$ be a list of all points in $\partial\,\mathbb{D}$. If $\beta \in I$ then the set

$$T(\beta) \overset{\text{def}}{=} \{\alpha \in I : \alpha \prec \beta \text{ and } w_\beta \in F(h_\alpha)\}$$

is at most countable. Then Corollary 3.3 shows that there exists a continuous curve $c_\beta : [0,\infty) \to \mathbb{D}$ in $\mathbb{D}$ ending tangentially at w_β and such that if $\alpha \in T(\beta)$ then

$$\lim_{s\to\infty} h_\alpha(c_\beta(s)) = (h_\alpha)_\flat(w_\beta)\,. \tag{3.2}$$

Define $\gamma(w_\beta) \overset{\text{def}}{=} c_\beta(0,\infty)$. We claim that for each $\alpha \in I$ the set $\mathrm{C}(h_\alpha,\gamma)$ is measurable and it has measure equal to 2π. Indeed, consider the set $F(h_\alpha)$ of Fatou points of h_α and consider its subset

$$S(\alpha) \overset{\text{def}}{=} \{w_\beta : \alpha \prec \beta \text{ and } w_\beta \in F(h_\alpha)\}$$

obtained by removing at most countably many points. Thus, $S(\alpha)$ is measurable and it has measure 2π. We claim that $S(\alpha) \subset \mathrm{C}(h_\alpha,\gamma)$. Indeed, if $w \in S(\alpha)$ then $w = w_\beta$ for some $\beta \in I$ such that $\alpha \prec \beta$ and $w_\beta \in F(h_\alpha)$. Thus, $\alpha \in T(\beta)$ and therefore (3.2) holds, i.e. $w = w_\beta \in \mathrm{C}(h_\alpha,\gamma)$. ☐

Definitions. A set has *small cardinality* if its cardinality is stricly less than the cardinality of the continuum. The *Baire space* $\mathbb{N}^\mathbb{N}$ is the collection of all sequences of natural numbers. Thus, $f \in \mathbb{N}^\mathbb{N}$ if and only if $f : \mathbb{N} \to \mathbb{N}$ is a sequence of natural numbers. The *dominating order* $\leq_*$ in the Baire space is an order relation defined as follows: $f \leq_* g$ if and only if there exists an integer m such that $f(n) \leq g(n)$ for each $n \geq m$.
We say that a model of ZFC has *Property D* if and only if for each $S \subset \mathbb{N}^\mathbb{N}$ of small cardinality there is a $g \in \mathbb{N}^\mathbb{N}$ such that $f \leq_* g$ for every $f \in S$.
We say that a model of ZFC has *Property* Unif $(\mathcal{N}) = \mathfrak{c}$ if and only if every subset of $\partial\,\mathbb{D}$ of small cardinality has Lebesgue measure zero.

Note. There are models of ZFC where both these properties hold but the Continuum Hypothesis does not hold; see Bartoszyński & Judah (1995).

3.3. **The statement of Theorem 1.2 holds in any model of ZFC having Properties D and Unif $(\mathcal{N}) = \mathfrak{c}$.**

Proof. The proof closely parallels the preceding one, the main difference being that, instead of a diagonal type argument (via the proof of Corollary 3.3) we use Property D. Let I be a set having the cardinality of the continuum and let $\prec$ be a well-ordering of I. Let $\{h_\alpha\}_{\alpha\in I}$ be a list of all bounded harmonic functions in $\mathbb{D}$ and let $\{w_\beta\}_{\beta\in I}$ be a list of all points in $\partial\,\mathbb{D}$. If $\beta \in I$ then the set

$$T(\beta) \overset{\text{def}}{=} \{\alpha \in I : \alpha \prec \beta \text{ and } w_\beta \in F(h_\alpha)\}$$

has small cardinality.
We claim that Theorem 3.2, and Property D imply that there exists a continuous curve $c_\beta : [0,\infty) \to \mathbb{D}$ in $\mathbb{D}$ ending tangentially at w_β and such that if $\alpha \in T(\beta)$ then (3.2) holds. Indeed, write $w_\beta = e^{is}$, and, for each $\alpha \in T(\beta)$, let $v_\alpha = (h_\alpha)_\flat(w_\beta)$ and define $f_\alpha \in \mathbb{N}^{\mathbb{N}}$ by letting $f_\alpha(n)$ be the smallest integer k such that

$$(h_\alpha)^*(s, 2e^{-\ell}; v_\alpha) \le \frac{1}{2^{n+n}}$$

for all $\ell \ge k$. Then the family $\{f_\alpha\}_{\alpha\in T(\beta)} \subset \mathbb{N}^{\mathbb{N}}$ has small cardinality. Property D implies that there is an element $f \in \mathbb{N}^{\mathbb{N}}$ such that $f_\alpha \le_* f$ for each $\alpha \in T(\beta)$. We may always assume that f is strictly increasing. The upper tangential curve $c = c_\beta$ ending at w_β with angle $\theta(\tau) = e^{-\tau}$ and such that $\frac{\theta(\tau)}{1-|c(\tau)|}$ interpolates linearly between 2^n and 2^{n+1} when τ is between $f(n)$ and $f(n+1)$ has the required property, by Theorem 3.2. Indeed, if $\alpha \in T(\beta)$ then there is a k such that if $n \ge k$ then $f_\alpha(n) \le f(n)$. Thus, if $n \ge k$ and $f(n) \le \tau < f(n+1)$ then $\frac{\theta(\tau)}{1-|c(\tau)|}(h_\alpha)^*(s, 2e^{-\tau}; v_\alpha) \le \frac{2}{2^n}$.
Define $\gamma(w_\beta) \stackrel{\text{def}}{=} c_\beta(0,\infty)$. We claim that for each $\alpha \in I$ the set $\mathrm{C}(h_\alpha, \gamma)$ is measurable and it has measure equal to 2π. Indeed, consider the set $F(h_\alpha)$ of Fatou points of h_α and consider its subset

$$S(\alpha) \stackrel{\text{def}}{=} \{w_\beta : \alpha \prec \beta \text{ and } w_\beta \in F(h_\alpha)\}$$

obtained by removing a certain set of small cardinality (thus a null set, because of our hypothesis on the model of ZFC we are working in). Thus, $S(\alpha)$ is measurable and it has measure 2π. We claim that $S(\alpha) \subset \mathrm{C}(h_\alpha, \gamma)$. Indeed, if $w \in S(\alpha)$ then $w = w_\beta$ for some $\beta \in I$ such that $\alpha \prec \beta$ and $w_\beta \in F(h_\alpha)$. Thus, $\alpha \in T(\beta)$ and therefore (3.2) holds, i.e. $w = w_\beta \in \mathrm{C}(h_\alpha, \gamma)$. □

4. Proof of Theorem 1.3

4.1. **Preliminaries.** The results of this section are analytic preliminaries holding in any model of ZFC. We do not assume that the tress is regular.

Notation. If $B \subset \partial\mathbb{D}$ then we denote by $1_B : \partial\mathbb{D} \to \{0,1\}$ the function equal to 1 on B and 0 on $\partial\mathbb{D}\setminus B$.

Lemma 4.1. *Assume that $B \subset \partial\mathbb{D}$ is open and that $m(\partial\mathbb{D}\setminus B) > 0$. Let γ be a tangential tress. Then for almost every $w \in \partial\mathbb{D}\setminus B$ the following holds:*

$$\liminf_{\substack{z\in\gamma(w)\\ z\to w}} P[1_B](z) = 0 \tag{4.1}$$

Proof. The proof is a variant of a technique used by Zygmund (1949), *mutatis mutandis.* For the benefit of the reader, we sketch the proof. Fatou's theorem implies that

$$\lim_{r\uparrow 1} P[1_B](rw) = 0 \tag{4.2}$$

An application of Egorov's theorem shows that for each $\epsilon > 0$ there is a perfect subset A of $\partial\mathbb{D}\setminus\{B\}$ such that the limit in (4.2) is uniform for $w \in A$ and $m(A) > 2\pi - m(B) - \epsilon$. We may assume that each $w \in A$ is a limit point of a sequence $we^{i\theta_n} \in A$ where $\theta_n \to 0$ and $\theta_n > 0$ for n even, $\theta_n < 0$ for n odd. It follows that (4.2) holds at each point $w \in A$, since $\{w\} \cup \gamma(w)$ is connected, and, therefore,

$\gamma(w)$ intersects the radii ending at $we^{i\theta_n}$ for an appropriate subsequence of $n's$, close enough to the boundary. The conclusion follows because ϵ is arbitrary. □

The *arc* in $\partial\mathbb{D}$ of *center* $e^{i\theta}$ and *radius* $r > 0$ is the subset of $\partial\mathbb{D}$ given by

$$\{e^{is} : \theta - r < s < \theta + r\}.$$

Definition. Let $\Gamma(w) \stackrel{\text{def}}{=} \{z \in \mathbb{D} : |z - w| < 2(1 - |z|)\}$, for each $w \in \partial\mathbb{D}$. Thus, Γ is a tress. If J is an arc in $\partial\mathbb{D}$, define

$$\triangle(J) \stackrel{\text{def}}{=} \{z \in \mathbb{D} : \Gamma^{\downarrow}(\{z\}) \subset J\}$$

The proof of the following basic and well known estimate is omitted.

Lemma 4.2. *There is a number $c_1 > 0$ such that*

$$P[1_J](z) \geq c_1$$

for each arc $J \subset \partial\mathbb{D}$ and each $z \in \triangle(I)$.

Definition. If $B \subset \partial\mathbb{D}$ is open and γ is a tress, then we define $Z_\gamma(B)$ as follows[6]: $w \in Z_\gamma(B)$ if and only if $w \in \partial\mathbb{D}\setminus\{B\}$ and there is a sequence J_k of arcs contained in B such that for all $k \in \mathbb{N}$

$$\gamma(w) \cap \triangle(J_k) \neq \emptyset$$

and for each $\epsilon > 0$ there is n_ϵ such that the set J_k is within the ball in $\mathbb{C}$ centered at w of radius ϵ, for $k \geq n_\epsilon$.

Note. The following result shows why the set $Z_\gamma(B)$ is of interest to us, and why we shall construct the open set B in such a way that (i) the set $Z_\gamma(B)$ is large and (ii) the set B has small measure.

Lemma 4.3. *Assume that $w \in Z_\gamma(B)$. Then*

$$\limsup_{\substack{z \to w \\ z \in \gamma(w)}} P[1_B](z) \geq c_1.$$

Proof. It follows from Lemma 4.2, since $P[1_J] \leq P[1_B]$ if $J \subset B$. □

4.2. **The Generalized Egorov Property.**

Definition. We say that the *Generalized Egorov Property* holds in a model of ZFC if the following statement holds: For every $\epsilon > 0$, every sequence of real valued functions defined on $\partial\mathbb{D}$ and converging pointwise to zero has a subsequence converging uniformly on a subset of $\partial\mathbb{D}$ whose outer measure is greater than $2\pi - \epsilon$.

Note. The functions in the previous statement are not necessarily measurable (when they are, Egorov's theorem yields a stronger conclusion, in any model of ZFC).

Theorem 4.4 (Weiss (2003))**.** *The Generalized Egorov Property is independent of ZFC.*

Note. In particular, there is a model of ZFC where the Generalized Egorov Property holds. The proof uses forcing; see Weiss (2003).

[6]This definition is inspired by the technique used in Zygmund (1949).

4.3. **The statement in Theorem 1.3 holds in any model of ZFC where the Generalized Egorov Property holds.**

Proof. Define the function $\tau : \partial\,\mathbb{D} \times \to (0,1]$ by $\tau(w,z) \overset{\text{def}}{=} \frac{1-|z|}{|w-z|}$ for $w \in \partial\,\mathbb{D}, z \in \mathbb{D}$. Let γ be a tangential tress and consider the sequence of everywhere defined functions $f_n : \partial\,\mathbb{D} \to (0,\infty)$ gauging the order of tangency at the various points:

$$f_n(w) \overset{\text{def}}{=} \sup\{\tau(w,z) : z \in \gamma(w), |z-w| < 2\pi/n\}\,. \tag{4.3}$$

Observe that $1 \geq f_n(w) \geq f_{n+1}(w)$ and that $\lim_{n\to\infty} f_n(w) = 0$ for each $w \in \partial\,\mathbb{D}$, since γ is tangential.
If $N \in \mathbb{N}$ then there is a set $C_N \subset \partial\,\mathbb{D}$ whose Lebesgue outer measure is greater than $2\pi - \frac{1}{2^N}$ and such that the sequence $\{f_n\}$ converges uniformly to 0 on C_N. We may and will assume that $C_N \subset C_{N+1}$ for all $N \in \mathbb{N}$. Thus, there is an element $\phi_N \in \mathbb{N}^{\mathbb{N}}$ such that

$$\text{if } \ell \in \mathbb{N} \text{ and } n \geq \phi_N(\ell) \quad \text{then } \sup_{w\in C_N} f_n(w) < 2^{-\ell}\,.$$

Define a strictly increasing sequence $\phi \in \mathbb{N}^{\mathbb{N}}$ dominating each ϕ_N, as follows. Let $\phi(1) \geq \phi_1(1)$, $\phi(2) \geq \max\{\phi_1(2), \phi_2(2)\}$, $\phi(3) \geq \max\{\phi_1(3), \phi_2(3), \phi_3(3)\}$, and so on. Then $\phi(i) \geq \phi_N(i)$ for all $i \geq N$.
It follows that

$$c(k) \overset{\text{def}}{=} \sup_{w\in C_k} f_{\phi(k)}(w) < 2^{-k}\,.$$

If $J \subset \partial\,\mathbb{D}$ is the arc $\{e^{is} : \theta - r < s < \theta + r\}$ of center $e^{i\theta}$ and radius $r >$ and $0 < c \leq 1$, then we denote $cJ \overset{\text{def}}{=} \{e^{is} : \theta - cr < s < \theta + cr\}$ the arc of center $e^{i\theta}$ and radius cr. Thus, $m(cJ) = cm(J)$.
For $n, p \in \mathbb{N}$ and $1 \leq p \leq n$ define $J(n,p) \overset{\text{def}}{=} \{e^{is} : (p-1)\frac{2\pi}{n} < s < p\frac{2\pi}{n}\} \subset \partial\,\mathbb{D}$.
Define

$$I_k \overset{\text{def}}{=} \bigcup_{p=1}^{\phi(k)} c(k)J(\phi(k),p)$$

Then

$$m(I_k) \leq 2\pi\, c(k) < 2\pi\, 2^{-k}\,.$$

Define

$$B(\ell) \overset{\text{def}}{=} \bigcup_{k=\ell}^{\infty} I_k\,.$$

Let

$$D \overset{\text{def}}{=} \bigcup_{1}^{\infty} C_N\,.$$

Then the outer measure of D is equal to 2π.

Claim. If $\ell_0 \in \mathbb{N}$ then $D \setminus B(\ell_0) \subset Z_\gamma(B(\ell_0))$.
If $h \in \mathrm{h}^\infty(\mathbb{D})$ and $w \in \partial\,\mathbb{D}$, we define

$$\operatorname{osc}(h; w) \overset{\text{def}}{=} \limsup_{\substack{z\to w\\ z\in\gamma(w)}} h(z) - \liminf_{\substack{z\to w\\ z\in\gamma(w)}} h(z)\,.$$

Consider $1_{B(\ell)} \in \mathrm{L}^\infty(\partial\mathbb{D})$ and its Poisson integral $P[1_{B(\ell)}] \in \mathrm{h}^\infty(\mathbb{D})$. Lemma 4.1 and Lemma 4.3 imply, in conjunction with the Claim, that there is a set $N(\ell)$ of Lebesgue measure zero such that if $w \in (D \setminus B(\ell)) \setminus N(\ell)$ then

$$\operatorname{osc}(P[1_{B(\ell)}]; w) \geq c_1 .$$

For $q > 1$ to be determined later, we define, following Zygmund (1949),

$$g \stackrel{\text{def}}{=} \sum_{\ell=1}^{\infty} q^{-\ell} 1_{B(\ell)} .$$

It follows that

$$P[g] = \sum_{\ell=1}^{\infty} q^{-\ell} P[1_{B(\ell)}] .$$

Define $N \stackrel{\text{def}}{=} \cup_1^\infty N(\ell)$. Then $m(N) = 0$. Define $B \stackrel{\text{def}}{=} \cap_1^\infty B(\ell)$. Then $m(B) = 0$. We now show that if $w \in (D \setminus B) \setminus N$ then $\operatorname{osc}(P[g]; w) > 0$. Indeed, let ℓ be the smallest integer n such that $w \notin B(n)$. Then w belongs to the open set

$$\bigcap_{k=1}^{\ell-1} B(k) \tag{4.4}$$

For $k = 1, 2, \ldots, \ell - 1$, the function $1_{B(k)}$ is equal to 1 on the set (4.4); since this set is open, it follows that for each $k = 1, 2, \ldots, \ell - 1$

$$\operatorname{osc}(P[1_{B(k)}]; w) = 0 .$$

On the other hand,

$$\operatorname{osc}(q^{-\ell} P[1_{B(\ell)}]; w) \geq q^{-\ell} c_1$$

and

$$\operatorname{osc}\Big(\sum_{k=\ell+1} q^{-k} P[1_{B(k)}]; w) \leq \sum_{k=\ell+1}^{\infty} q^{-k} \leq q^{-\ell} \frac{1}{q-1} .$$

It follows that

$$\operatorname{osc}(P[g]; w) \geq q^{-\ell} c_1 - q^{-\ell} \frac{1}{q-1} > 0$$

if q is chosen greater than $\frac{1+c_1}{c_1}$.
Since the set $(D \setminus B) \setminus N$ has outer measure equal to 2π, the proof is completed. □

Proof of the Claim. Assume that $w_0 \in D \setminus B(\ell_0)$. The set $\gamma(w_0)$ contains a branch ending tangentially at w_0 from one side. Assume it ends at w_0, say, from the right. Let $N_0 \in \mathbb{N}$ be such that $w_0 \in C_{N_0}$. Let $\rho_0 > 0$ be such that if $z \in \gamma(w_0)$ and $|z - w_0| < \rho_0$ then $\tau(w_0, z) < 2^{-10}$. Choose $z_0 \in \gamma(w_0)$ such that $|z_0 - w_0| < \rho_0$. Choose $\ell_1 \in \mathbb{N}$ such that $\ell_1 \geq \ell_0$, $\ell_1 \geq N_0$ and

$$\frac{2\pi}{\phi(\ell_1)} < 2^{-10} |w_0 - z_0| .$$

Let $\ell \geq \ell_1$. Then $w_0 \notin B(\ell)$. Let $k \geq \ell$. Then $w_0 \notin I_k$. Let $p \in \{1, 2, \ldots, \phi(k)\}$ be such that the arc

$$J_k \stackrel{\text{def}}{=} c(k) J(\phi(k), p)$$

is closer to w from the right. We know that $w_0 \in C_k$, since $k \geq N_0$. Thus,

$$\sup \left\{ \tau(w_0, z) : z \in \gamma(w_0), |z - w_0| < \frac{2\pi}{\phi(k)} \right\} \leq c(k) .$$

Let w_1 be the center of the arc J_k. Then there is a point $z_1 \in \gamma(w_0)$ such that

$$|z_1 - w_0| = |w_1 - w_0|$$

and z_1 is located on the same side as $\gamma(w_0)$. Observe that $|w_1 - w_0| < \frac{2\pi}{\phi(k)}$. It follows that $\tau(w_0, z_1) \leq c(k)$. Thus, $z_1 \in \triangle(J_k)$.

5. Proof of Theorem 1.4

The proof follows the same scheme of the proof of Theorem 1.3. The main observation is that now all the functions and sets involved are measurable. Indeed, the functions f_n defined in (4.3) are measurable, because the tress is regular. We leave the details to the reader.

References

Aikawa, H. 1990. Harmonic functions having no tangential limits. *Proc. Amer. Math. Soc.*, **108**, 457–464.

Aikawa, H. 1991. Harmonic functions and Green potentials having no tangential limits. *J. London Math. Soc. (2)*, **43**, 125–136.

Bartoszyński, T., & Judah, H. 1995. *Set Theory. On the Structure of the Real Line.* A K Peters.

Boehme, T.K.., & Weiss, M.L. 1971. Extensions of Fatou's theorem to tangential asymptotic values. *Proc. Amer. Math. Soc.*, **27**, 289–298.

Cohen, P.J. 1966. *Set Theory and the continuum hypothesis.* Benjamin.

Di Biase, F. 1998. *Fatou Type Theorems. Maximal Functions and Approach Regions.* Birkhäuser.

Di Biase, F., Stokolos, A., Svensson, O., & Weiss, T. 1998. Tangential boundary behaviour of bounded harmonic functions in the unit disc. *Geometry Seminars 1996-97*, 63–68.

Drake, F.R. 1974. *Set Theory.* North-Holland.

Fatou, P. 1906. Séries trigonométriques et séries de Taylor. *Acta Math.*, **30**, 335–400.

Fefferman, C., & Stein, E.M. 1971. Some maximal inequalities. *Amer. J. Math.*, **93**, 107–115.

Jech, T. 1978. *Set Theory.* Academic Press.

Kunen, K. 1980. *Set Theory. An Introduction to Independence Proofs.* North-Holland.

Littlewood, J.E. 1927. Mathematical notes (4): On a theorem of Fatou. *J. London Math. Soc.*, **2**, 172–176.

Loomis, L.H. 1943. The converse of the Fatou theorem for positive harmonic functions. *Trans. Amer. Math. Soc.*, **53**, 239–250.

Lusin, N., & Sierpiński, W. 1917. Sur une dècomposition d'un intervalle en une infinitè non dènombrable d'ensembles non mesurables. *C. R. Académie des Sciences*, **165**, 422–424.

Nagel, A., & Stein, E.M. 1984. On certain maximal functions and approach regions. *Adv. Math.*, **54**, 83–106.

Rudin, W. 1979. Inner function images of radii. *Math. Proc. Cambridge. Philos. Soc.*, **85**(2), 357–360.

Rudin, W. 1988. Tangential H^∞-images of boundary curves. *Math. Proc. Cambridge. Philos. Soc.*, **104**(1), 115–118.

Stein, E.M. 1993. *Harmonic Analysis: real-variable methods, orthogonality, and oscillatory integrals.* Princeton University Press.

Stein, E.M., & Weiss, G.L. 1971. *Introduction to Fourier analysis on Euclidean spaces.* Princeton University Press.

Weiss, T. 2003. *A note on generalized Egorov's theorem.* Preprint, Submitted.

Zygmund, A. 1949. On a theorem of Littlewood. *Summa Brasil. Math.*, **2**(5), 51–57.

October 14, 2003 h 16: 52 min

Dipartimento di Scienze, Università 'G. d'Annunzio', Viale Pindaro 87, 65127 Pescara, Italy
E-mail address: f.dibiase@unich.it

DePaul University, Department of Mathematical Sciences, 2320 North Kenmore Ave. Chicago IL 60614, U.S.A.
E-mail address: astokolo@depaul.edu

Department of Science and Technology, Campus Norrköping, Linköping University, 60174 Norrköping, Sweden
E-mail address: olosv@itn.liu.se

Institute of Mathematics, Akademia Podlaska, 08-110 Siedlce, Poland
E-mail address: tomaszweiss@go2.pl