STATEMENT OF RESEARCH INTERESTS

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1 Selected past research

1.1 Differentiation of integrals in $\mathbb{R}^2$ by bases of intervals. A. Zygmund's program.

Let

$$M_B f(x) = \sup_{B \ni I \ni x} |I|^{-1} \int_I |f(y)|\, dy$$

where B is a *differentiation basis*, a collection of sets that are somehow natural for the given problem.

A. Zygmund initiated the study of maximal functions associated with a differentiation basis of some multi-dimensional intervals associated with various groups of dilations. This is called in literature "Zygmund's program".

I was able to solve Zygmund's program in the two-dimensional case [39].

More specifically, if the dilations given by $\delta_t(x_1, ..., x_n) = (tx_1, ..., tx_n)$ naturally defines the basis to be the basis of all possible cubes in $\mathbb{R}^n$. The corresponding maximal operator is called Hardy-Littlewood maximal operator. It has weak type (1,1).

On the other hand, the dilations given by $\delta_{t_1,...,t_n}(x_1, ..., x_n) = (t_1x_1, ..., t_nx_n)$ naturally defines the basis to be the basis of all possible n-dimensional intervals in $\mathbb{R}^n$ The corresponding maximal operator is called strong maximal operator. It has weak type $L(\log^+ L)^{n-1}$ which is non-improvable.

Since the basis of cubes is a sub-basis of the basis of rectangles, the rarefaction of the basis of rectangles can improve the weak type estimate, potentially makes a basis with the intermediate $L\sqrt{\log^+ L}$ weak type estimate in $\mathbb{R}^2$. In particular, Zygmund conjectured that a collection of rectangles with the sides parallel to the coordinate axis of dimensions $s \times t$ such that $0 < s \leq 1, 0 < s^2 \leq t \leq s$ is a such type basis.

It turns out that *every maximal function associated with the translation invariant basis of rectangles in $\mathbb{R}^2$ behaves as either a maximal function, associated with a basis consisting of all squares and mapping L^1 into weak L^1, or a maximal function, associated with all rectangles and mapping $L\log^+ L$ into weak L^1, and these classes cannot be improved.* I consider this alternative phenomenon extremely interesting and quite unexpected. Moreover, I found a simple geometric characteristic that determines which class, L^1 or $L\log^+ L$, needs to be chosen.

1.2 Tauberian Condition of Córdoba-Fefferman and L^p boundedness of geometric maximal operators (with P. Hagelstein).

Córdoba and Fefferman in [8] introduced a *Tauberian condition* for the maximal operator M_B if there exists a finite constant C such that for any measurable set $E \subset \mathbb{R}^2$ one has the inequality

$$\left|\left\{x : M_B\chi_E(x) > \frac{1}{2}\right\}\right| \leq C|E|.$$

In [8] it was characterized as a very weak type restriction on the basis B.

In joint work with Hagelstein [22] we discovered that for the most important case of homothecy invariant bases B in $\mathbb{R}^n$ the condition is quite strong. *It implies L^p-boundedness of M_B for large p.* Recently, Bateman [4] prove that in the case of directional maximal function in $\mathbb{R}^2$ it implies the L^p-boundedness of M_B for all $1 < p \leq \infty$.

1.3 Differentiation of integrals in $\mathbb{R}^2$ by bases of convex sets.

It was noticed by A. Zygmund that the basis of all convex sets does not differentiate the characteristic functions of certain open and closed sets. Since that time there has been little interest in the basis of arbitrary rectangles due to its bad properties. However, I found [38] that the investigation of this basis is not only interesting, but also reasonable.

It turns out that if we partition the L^p-scale using the degree of integral smoothness, the basis of arbitrarily oriented rectangles has quite convenient properties. *It turns out that if the modulus of continuity in $L^1(\mathbb{T}^2)$ norm satisfies the estimate $\omega(f,h) = O(h), h \to +0$ then the integral of f is differentiated by the basis of convex sets.* So, the best first order smoothness of functions guarantees the differentiation of integrals in a given sense. It is remarkable that, firstly, *this result is in essence two-dimensional* and, secondly, *no other smoothness is sufficient for this.* Recently, H. Aimar, L. Forzani and V. Naibo [1, 31] have found a reasonable extension of this result to the multidimensional settings.

1.4 Tangential Fatou property. The problem of W. Rudin (with F. diBiase, O. Svensson and T. Weiss).

The famous theorems of Fatou and Littlewood say that the bounded harmonic functions in the unit disc D in the complex plane converge nontangentially almost everywhere and fail to converge along the rotates of any given tangential curve. W. Rudin asked a question whether exists a family of tangential curves whose shape vary from point to point such that every bounded harmonic function has a limit a.e. along that curves. The shocked answer is *undecidable.* It was established by F.Di Biase, O. Swensson, T. Weiss and myself in [16]. The proof is based on delicate geometrical and analytical considerations combined with the techniques of modern logic.

1.5 A_∞ via Gurov-Reshetnyak condition (with A. Korenovskiy and A. Lerner).

The famous A_∞ is widely in use in modern Harmonic Analysis. There are few important characterizations of this conditions found by R. Coiffman and Ch. Fefferman [9]. In particular,

$$A_\infty = \bigcup_{p>1} A_p = \bigcup_{q>1} RH_q$$

where A_p denotes Muckenhoupt's class and RH_q denotes class of functions that satisfies the reverse Hölder inequality. All the known characterization are highly non-linear. A. Korenovskiy, A. Lerner and myself [26] has found an almost linear form of A_∞ condition, proved that

$$A_\infty = \bigcup_{0<\varepsilon<2} GR_\varepsilon$$

where GR_ε denotes class of nonegative functions that satisfies Gurov-Reshetnyak condition

$$\frac{1}{|Q|}\int_Q |f(x) - f_Q|dx \le \varepsilon f_Q, \qquad f_Q = \frac{1}{|Q|}\int_Q f(x)dx.$$

1.6 F. Riesz Sun Rising Lemma in $\mathbb{R}^n$ (with A. Korenovskiy and A. Lerner).

Famous Riesz Sun Rising Lemma allows to prove many one-dimensional inequalities with the best constants. The proof heavily depends on the linear structure of open sets in $\mathbb{R}$ and cannot be translated to $\mathbb{R}^n$. A. Korenovskiy, A. Lerner and myself [27] proposed a multidimensional version of Riesz Lemma, which allows to generalized several classical one-dimensional results to $\mathbb{R}^n$ (for details see [24]).

1.7 Maximal functions of Fourier multiplier operators (with W. Trebels).

Let $m \in L^\infty(0, +\infty)$ and denote by $m_t\,, t > 0\,$, the function $m_t(u) = m(tu)\,,\, u > 0\,$; define operators $T_{m_t}\,(t > 0)$ on $L^2(R^n)$ via their Fourier transform

$$(\widehat{T_{m_t}f})(\xi) = m_t(|\xi|)\hat{f}(\xi).$$

Note that the Abel-Poisson means are defined by $m(u) = e^{-u}$, the Gauss-Weierstrass means by $m(u) = e^{-u^2}$, the Bochner-Riesz means by $m(u) = (1-u^2)_+^\alpha$. These means are very important in the problems of Harmonic Analysis, Partial Differential Equations, Theory of Probability etc. – convergence a.e. of the Poisson and the Gauss-Weierstrass means is one of the basic facts of Harmonic Analysis. We mention that the important problem of a.e. convergence of the Bochner-Riesz means is only partially solved when $n \geq 2$.

The question of convergence a.e. is related to the problem of the boundedness of the corresponding maximal operator

$$T_m^* f(x) = \sup_{t>0} T_{m_t} f(x).$$

In this general setting not much investigations were done. We can mention papers of A. Carbery [5], H. Dappa and W. Trebels [14] and recent papers of M. Crist, L. Grafakos, P. Honzík, and A. Seeger [10, 19].

A. Kamaly, W. Trebels and myself in [25, 40, 41] found the application for the maximal multiplier theorems considerind the following problem: *Under which smoothness conditions on f does $T_{m_t}f$ converge a.e. towards f with a prescribed rate $w(t)$ of convergence?*

1.8 Monge-Ampere PDE and the Bellman function of dyadic maximal operator (with L. Slavin and V. Vasunin).

The following Bellman function $\mathcal{B}$ was introduce by F. Nazarov and S. Treil [32] for the diadic maximal function $M\varphi(x)$

$$\mathcal{B}(f, F, L) = \sup_{0 \leq \varphi \in L^p_{\text{loc}}(\mathbb{R})} \left\{ \langle (M\varphi)^p \rangle_Q : \langle \varphi \rangle_Q = f; \langle \varphi^p \rangle_Q = F; \sup_{R \supset Q} \langle \varphi \rangle_R = L \right\}.$$

Here Q and R are diadic intervals on the line and $\langle\varphi\rangle_Q = |Q|^{-1}\int_Q \varphi(x)\,dx$. The function $\mathcal{B}$ is independent of Q and the domain of $\mathcal{B}$ is $\Omega = \{(f, F, L) : 0 \le f \le L; f^p \le F\}$. It was shown in [32] that

$$\mathcal{B}(f, F, L) \le pL^p - pqL^{p-1}f + q^pF \tag{1}$$

where $1/p + 1/q = 1$. It gives us the inequality

$$\frac{1}{|J|}\int_J (Mf)^p \le \frac{q^p}{|J|}\int_J f^p - \frac{1}{(p-1)}\left(\frac{1}{|J|}\int_J f\right)^p$$

which even a little bit better then classical Doob's inequality.

The exact Bellman function $\mathcal{B}(f, F, L)$ was computed in a brilliant paper of Melas [30], who managed to follow through on the original Bellman setup in [32] and find the dyadic Bellman function. It turns out to be only C^1 function. Namely, let $H(z) = -(p-1)z^p + pz^{p-1}$, $H : [1,\infty) \to (-\infty, 1]$ and $\omega : (-\infty, 1] \to [1,\infty)$ be the inverse function to H. In these terms

$$\mathcal{B}(f, F, L) = \begin{cases} F\omega\left(\frac{pfL^{p-1}-(p-1)L^p}{F}\right)^p & L < qf \\ L^p + q^p(F - f^p) & L \ge qf \end{cases}$$

What is unique, however, about Melas's approach is that he *does not solve the Bellman PDE (nor does he establish what it may be)* but instead relies on the geometrical properties of the operator M combined in a quite sophisticated way with combinatorial and analytical considerations. Slavin, Stokolos and Vasunin [37] were able to find the Bellman PDE for $\mathcal{B}$ and get the function $\mathcal{B}(f, F, L)$ as a solution of that PDE.

$\mathcal{B}(f, F, L) = L^p\mathcal{B}(f/L, F/L^p, 1)$ and let $G(x, y) = \mathcal{B}(f/L, F/L^p, 1)$ where $x = f/L, y = F/L^p$.

Now, it is enough to find $G(x, y)$, which is a function of only two variables.

Slavin, Stokolos and Vasunin [37] were able to find the Bellman PDE for $\mathcal{B}$. It turnes out that the Bellman function came up as a solution of the following Monge-Ampere boundary value problems:

$$(*) \qquad \begin{cases} G_{xx}G_{yy} = G_{xy}^2 \quad \text{for } \ 0 \le x \le 1;\ x^p \le y \\ \\ G(x, x^p) = 1, \quad pG(1, y) = G_x(1, y) + pyG_y(1, y) \end{cases}$$

and

$$(**) \qquad \begin{cases} G_{xx}G_{yy} = G_{xy}^2 \quad \text{for } \ 0 \le x \le 1;\ x^p \le y \\ \\ G(x, x^p) = 1, \quad G(0, y) = 1 + q^py \end{cases}$$

whose solutions give us two different functions.

The first one is

$$G(x,y) = y\omega\left(\frac{px-(p-1)}{y}\right)^p.$$

The second one has solution

$$G(x,y) = 1 + q^p(y - x^p)$$

To check that these functions indeed satisfies (*) and (**) is a simple direct computation.

In terms of f, F and L they might be written as

$$B(f,F,L) = L^pG(f/L, F/L^p) = F\omega\left(\frac{pfL^{p-1}-(p-1)L^p}{F}\right)^p.$$

and

$$B(f,F,L) = L^p + q^p(F - f^p).$$

correspondingly. These are parts of Melas function.

2 Ongoing projects

2.1 Hypercontractivity of a certain semigroups of operators (with L. De Carli and W. Urbina)

The theory of semigroups of operators is a crossroad of different areas of Mathematics, among which we can mention PDE, functional analysis, harmonic analysis, the theory of orthogonal polynomials and special functions, probability, and control theory.

A semigroup $\{P_t\}$ is *hypercontractive* if for each initial condition $1 < p < \infty$ there exists a strictly increasing function $q : \mathbb{R}^+ \to [p, \infty)$, such that, for every $t > 0$,

$$||P_t f||_{L^{q(t)}(d\mu)} \leq ||f||_{L^p(d\mu)}, .$$

whenever $f \in L^p(d\mu)$. This definition can be also extended to complex values of t, and indeed the complex valued case has numerous and deep applications.

The hypercontractivity problem for the Ornstein-Uhlenbeck semigroup has been completely solved for real and complex values of t, but for other classical semigroups there are still many open problems.

The Ornstein-Uhlenbeck semigroup $\{T_t\}$ in $\mathbb{R}^d$ is a positive conservative symmetric diffusion semigroup, strongly L^p-contractive for $1 \leq p \leq \infty$. Its infinitesimal generator is the Ornstein-Uhlenbeck operator,

$$Lf(x) = \frac{1}{2}\sum_{i=1}^{d}[\frac{\partial^2 f}{\partial x_i}(x) - x_i \frac{\partial f}{\partial x_i}(x)] = \frac{1}{2}\Delta f(x) - x \cdot \nabla f(x),$$

with invariant measure the gaussian measure $\gamma_d(x)$.

L. Gross [21] proved that the hypercontractive property is equivalent to the fact that the Ornstein-Uhlenbeck operator satisfies the *(tight) logarithmic Sobolev inequality*: for any $f \in L^2(\gamma_d)$ with ∇f, in weak sense, belonging to $L^2(\gamma_d)$

$$\int_{\mathbb{R}^d} |f(x)|^2 \log |f(x)| \gamma_d(dx) \leq \frac{1}{2}\int_{\mathbb{R}^d} |\nabla f(x)|^2 \gamma_d(dx) + ||f||^2_{2,\gamma_d} \log ||f||_{2,\gamma_d}, \quad (2)$$

The logarithmic Sobolev inequality generalizes, for the gaussian measure, the classical *Sobolev inequality* : for any function $f \in L^2(\mathbb{R}^d)$ with $\nabla f \in L^2(\mathbb{R}^d)$, in weak sense, then $f \in L^p(\mathbb{R}^d)$ for $p^{-1} = (\frac{1}{2} - \frac{1}{d})$, and moreover,

$$||f||_p \leq C_d \int_{\mathbb{R}^d} |\nabla f(x)|^2 dx.$$

The Gaussian measure can be defined in spaces of infinite dimension and as the logarithmic Sobolev inequality is independent of dimension, it can be extended to this context too.

One important application of the hypercontractive property of Ornstein-Uhlenbeck semigroup is in the multiplier theory for Hermite expansions.

The generalized Hermite polynomials were defined by G. Szëgo in [34] (see problem 25, p.380) as being orthogonal polynomials with respect to the measure $d\lambda(x) = d\lambda_\mu(x) = |x|^{2\mu}e^{-|x|^2}dx$, with $\mu > -1/2$.

The generalized Hermite polynomial of degree n and type μ can be defined using the Laguerre polynomials L_m^γ as follows: for n even

$$H_{2m}^\mu(x) = (-1)^m(2m)!\frac{\Gamma(\mu+\frac{1}{2})}{\Gamma(m+\mu+\frac{1}{2})}L_m^{\mu-\frac{1}{2}}(x^2) \tag{3}$$

and for n odd

$$H_{2m+1}^\mu(x) = (-1)^m(2m+1)!\frac{\Gamma(\mu+\frac{3}{2})}{\Gamma(m+\mu+\frac{3}{2})}xL_m^{\mu+\frac{1}{2}}(x^2), \tag{4}$$

It can be proved that generalized Hermite polynomial satisfies the following differential equation

$$(H_n^\mu)''(x) + 2(\frac{\mu}{x} - x)(H_n^\mu)'(x) + 2(n - \mu\frac{\theta_n}{x^2})H_n^\mu(x) = 0, \tag{5}$$

with

$$\theta_n = \begin{cases} 1 & \text{if} \quad n \text{ is odd,} \\ 0 & \text{if} \quad n \text{ is even.} \end{cases}$$

and $n \geq 0$. Therefore, by considering the (differential-difference) operator

$$L_\mu = \frac{1}{2}\frac{d^2}{dx^2} + (\frac{\mu}{x} - x)\frac{d}{dx} - \mu\frac{I-\tilde{I}}{2x^2}, \tag{6}$$

where $If(x) = f(x)$ and $\tilde{I}f(x) = f(-x)$, H_n^μ turns out to be an eigenfunction of L_μ with eigenvalue $-n$. This operator is one example of a Dunkl operator in one dimension. The theory of Dunkl operators originated in [13] and it is nowadays a very rich theory.

Let us consider then the semigroup $\{T_\mu^t\}$ associated with $\{H_n^\mu\}$,

$$T_\mu^t f(x) = \int_{-\infty}^{\infty} p_\mu(t,x,y)f(y)\lambda(dy),$$

where $p_\mu(t,x,y) = \sum_{n=0}^\infty \frac{\gamma_\mu(n)}{2^n(n!)^2}H_n^\mu(x)H_n^\mu(y)e^{-nt}$.

The family of operators $\{T_\mu^t\}_{t\geq 0}$ is then a semigroup of operators with generator L_μ, that we will call the *generalized Ornstein-Uhlenbeck semigroup*. For $\mu = 0$, $\{T_\mu^t\}$ reduces to the Ornstein-Uhlenbeck semigroup. By using the generalized Mehler's formula for x, $y \in \mathbb{R}$ and $|z| < 1$,

$$\sum_{n=0}^\infty \frac{\gamma_\mu(n)}{2^n(n!)^2}H_n^\mu(x)H_n^\mu(y)z^n = \frac{1}{(1-z^2)^{\mu+1/2}}e^{-\frac{z^2(x^2+y^2)}{1-z^2}}e_\mu\left(\frac{2xyz}{1-z^2}\right). \tag{7}$$

we can obtain the following integral expression of this generalized Ornstein-Uhlenbeck semigroup $\{T_\mu^t\}$,

$$T_\mu^t f(x) = \frac{1}{(1-e^{-2t})^{\mu+1/2}} \int_{-\infty}^{\infty} e^{-\frac{e^{-2t}(x^2+y^2)}{1-e^{-2t}}} e_\mu\left(\frac{2xye^{-t}}{1-e^{-2t}}\right) f(y)|y|^{2\mu} e^{-|y|^2} dy. \qquad (8)$$

It is not known whether T_t^μ is an hypercontractive semigroup or not. Because of the relation between generalized Hermite polynomials and Laguerre polynomials (3), the natural conjecture is that in fact T_μ^t is hypercontractive. We plan to work on this problem.

2.2 Sharpness of the ergodic Danford-Zygmund theorem (with J. Rosenblatt).

Probably, it is not a surprise that there is strong relation between convergence of integral means (i.e. differentiation of integrals) and convergence of ergodic means. In particular, Danford [12] and Zygmnud [44] establish an ergodic version of the Jessen-Marcinkiewicz-Zygmnud [23] theorem.

Theorem[12, 44] *Let U, V be one-to-one measure-preserving maps of a measure space Ω of finite measure onto itself, and let $f \in L\log^+ L(\Omega)$. Then*

$$\frac{1}{mn}\sum_{k=0}^{n-1}\sum_{l=0}^{m-1} f(U^kV^lx) \quad \text{a.e. convergent, as } \min(m,n) \to \infty$$

The proof of the JMZ theorem based on the summability of a certain maximal functions. Fava [17] made precise the class of function to to which the the multidimensional pointwise ergodic theorem applies. In two-dimensional case one needs to assume about f that $\sup_n |\sum_{k=1}^n f(T^kx)| \in L$.

However, when one tries to prove the the Danford-Zygmund-Fava theorem for integrable functions, then the control of the intermediate maximal function fails. Indeed, it is actually the case that the multiple averages will not converge pointwise a.e. Results about the failure of a.e. convergence when dealing with iterated processes are what is essentially being addressed in Al-Husaini [2], Blackwell-Dubins [11], Derrinennic-Lin [15] and Kachurovskii [24].

These examples demonstrates that the property of summability of a certain maximal functions are crucial in this setting. Stein's phenomena state that for the Hardy-Littlewood maximal function Mf and any cube Q, $Mf \in L(Q)$ if and only if $f \in L\log^+ L(Q)$. This result was transfer to the ergodic settings by Ornstein [33]. In [22, **?**] Hare, Stokolos and Hagelstein found a necessary and sufficient condition for the Stein's phenomena in case of rare maximal function. Argiris and Rosenblatt [3] has studied convergence questions of the sequences T_nf_n for operator sequences and suppremum $\sup_n |f_n|$ that is not integrable. They rased up a question about sharpness of the Danford-Zygmund-Fava Theorem, proving the following

Theorem[3] *(i) If U and V being a power of V then convergence has to be true for all $f \in L^1(\Omega)$.*

(ii) If V is any ergodic invertible transformation and f is any positive function that is not in $L\log^+ L$ then there exists another ergodic transformation U such that convergence is not longer happen.

In the Theorem **??** it mighty be not difficult to replace the hypothesis "*U and V being a power of V*" with the condition

$$\exists (i,j) \in \mathbb{Z}^2 \text{ such that } \mu\{x : U^i V^j x \neq x\} = 0 \tag{9}$$

On the other hand, there is an approach that allows one to prove that $L\log^+ L$ condition in Theorem 2.2 is sharp provided non-periodicity of commuting measure preserving transformations, i.e. that

$$\forall (i,j) \in \mathbb{Z}^2 \; \mu\{x : U^i V^j x = x\} = 0 \tag{10}$$

Conjecture *Let $\phi(x) = o(\log(x))$ where $x \to \infty$ and U and V are pair of commuting invertible non-periodic m.p.t.s on Ω. Then there exist a function $f \in L\phi(L)(\Omega)$ such that*

$$\frac{1}{mn}\sum_{k=0}^{n-1}\sum_{l=0}^{m-1} f(U^k V^l x) \quad \textit{fail to convergent a.e. , as } \min(m,n) \to \infty$$

A very interesting question is to verify what's going on when neither (9) nor (10) is longer true. It might be that this way one can construct a pair of m.p.t. U and V such that the Theorem A will be valid for $L\sqrt{\log^+ L}$ function and not better - something that Zygmund was very interested to know. Especially, having in mind that it is impossible to do with the integral means. Let us be more specific about this in the following subsection.

2.3 Regularity of the data and the boundary behavior of the solutions of PDE (with V. Krotov).

Suppose that Ω is a bounded Lipschitz domain in $\mathbf{R}^n, n \geq 2$, μ is the Lebesgue measure on $\partial\Omega$,

$$\Gamma^*_\varepsilon(P) = \{x \in \Omega : |x - P| < a(\text{dist}\,(x, \partial\Omega))^\varepsilon\},\; P \in \partial\Omega,$$

and

$$N_\varepsilon u(P) = \sup\{|u(x)| : (x) \in \Gamma^*_\varepsilon(P)\}$$

is the corresponding maximal operator.

For $\varepsilon = 1$, the domain $\Gamma^*_\varepsilon(P)$ admits a nontangential approach to the boundary: if a is sufficiently large (greater than the Lipschitz constant of the domain Ω), then $\Gamma^*_1(P)$ contains a cone of fixed altitude and angle independent of $P \in \partial\Omega$. But if $\varepsilon < 1$, then the domain $\Gamma^*_\varepsilon(P)$ a tangential approach to the boundary; moreover, the degree of tangency of $\Gamma^*_\varepsilon(P)$ to $\partial\Omega$ increases as ε decreases.

Let $p > 0$ and $m \in \mathbf{N}$. Introduce Hardy-Sobolev classes

$$\mathcal{H}^p_m(\Omega) = \left\{ u \in C^m(\Omega) : \|u\|_{\mathcal{H}^p_m} = \sum_{l=0}^{m} \|N_1(\nabla^l u)\|_{L^p_\mu(\partial\Omega)} < \infty \right\}.$$

Since

$$\|N_1 u\|_{L^p_\mu(\partial\Omega)} \le c \left(|u(0)| + \|N_1(\nabla u)\|_{L^p_\mu(\partial\Omega)} \right)$$

then

$$\|u\|_{\mathcal{H}^p_m(\Omega)} \asymp \sum_{l=0}^{m-1} \left|\nabla^l u(0)\right| + \|N_1(\nabla^m u)\|_{L^p_\mu(\partial\Omega)}.$$

Theorem[28]. *Suppose $m \in \mathbf{N}$, $0 < p < (n-1)/m$, $(n - mp - 1)/(n-1) = \varepsilon$. Then for any function $u \in \mathcal{H}^p_m(\Omega)$ the following inequality is valid:*

$$\|N_\varepsilon u\|_{L^p(\partial\Omega)} \le c \|u\|_{\mathcal{H}^p_m(\Omega)}$$

and for almost all $P \in \partial\Omega$ the limit $\Gamma_\varepsilon - \lim_{x\to P} u(x)$ exists.

For instance, the following sets of parameters satisfy the conditions of the theorem: $n = 4, p = 2, m = 1, \varepsilon = 1/3$.

Since the solutions of the Dirichlet problem for the homogeneous real constant coefficient elliptic PDE

$$\begin{cases} \mathcal{L}u = 0 & \text{in } \Omega, \\ N_1(\nabla^{m-1} u) \in L^p(\partial\Omega) \end{cases}$$

and

$$\begin{cases} \mathcal{L}u = 0 & \text{in } \Omega, \\ N_1(\nabla^{m} u) \in L^p(\partial\Omega) \end{cases}$$

belongs to $\mathcal{H}^p_{m-1}(\Omega)$ and $\mathcal{H}^p_m(\Omega)$ correspondingly one can claim the existence of tangential boundary values of the solutions.

Moreover, *it is possible to control the rate of convergence.*

It might by proved that for each function $u \in \mathcal{H}^p_m(\Omega)$ all partial derivatives $D^k u$ of order $|k| < m$ have nontangential limits (denoted by $D^k u(P)$) almost everywhere on $\partial\Omega$. Theorefore, if $l < m$, then for each function $u \in \mathcal{H}^p_m(\Omega)$ and almost all $P \in \partial\Omega$ the expression $T_l(x, P; u)$ is meaningful, where

$$T_l(x, y; u) = u(x) - \sum_{|k|<l} \frac{(x-y)^k}{k!} D^k u(y)$$

the Taylor remainders of the function $u \in C^l(\Omega)$. Let us introduce the notation

$$W_\sigma u(x, P) = \frac{T_{[\sigma]}(x, P; u)}{|x - P|^\sigma}.$$

It follows from the above that for $\sigma < m$ this expression is well defined for $u \in \mathcal{H}^p_m(\Omega)$ and $x \in \Omega$ for almost all $P \in \partial\Omega$.

The above theorem can be regarded as the limiting particular case $\sigma = 0$ of the following theorem.

Theorem[28]. *Suppose that $p > 0$, $m \in \mathbf{N}$, $0 < \sigma < m < (n-1)/p$, $(n - mp - 1)/(n - \sigma p - 1) = \varepsilon$. Then for any function $u \in \mathcal{H}^p_m(\Omega)$ the following inequality is valid:*

$$\|N_\varepsilon(W_\sigma u)\|_{L^p_\nu(\partial\Omega)} \leq c\|u\|_{\mathcal{H}^p_m(\Omega)}$$

and for almost all $P \in \partial\Omega$

$$\Gamma_\varepsilon - \lim_{x\to P} \frac{T_{[\sigma]}(x, P; u)}{|x-P|^\sigma} = 0.$$

For instance, the following sets of parameters satisfy the conditions of the theorem: $\sigma = 1, n = 6, p = 2, m = 2, \varepsilon = 1/3$. In this case $T_1(x, P, u) = u(x) - u(P) - (x-P)\cdot\nabla u(P)$ and along the tangential region $\Gamma^*_{1/3}(P)$

$$u(x) - u(P) = (x-P)\cdot\nabla u(P) + o_P(1)(|x-P|) \qquad a.e.$$

What happen if $mp > n-1$? In this case there is an integer l such that $l \leq m - (n-1)/p < l+1$

Theorem[28]. *Suppose that $p > 0$, $m \in \mathbf{N}$, $m \geq (n-1)/p$, and l as above. Then if $u \in C^m(\Omega)$, $N_1(\nabla^m u) \in L^p(\partial\Omega)$, then:*

1). If $l = m - (n-1)/p$ then almost all $P \in \partial\Omega$ the quantity $\nabla^l u$ has a limit along the regions of exponential rate of tangency

$$\left\{x \in \Omega : |x-P| < a\left(\ln\frac{e\mathrm{diam}(\Omega)}{\mathrm{dist}(x,\partial\Omega)}\right)^{-\frac{q-1}{n-1}}\right\}, \quad \frac{1}{q} = \frac{1}{p} - \frac{m-l-1}{n-1};$$

In particular, if $p = n-1 > 1$ the condition $N_1(\nabla u) \in L^p(\partial\Omega)$ implies that u has limit along the domains

$$\left\{x \in \Omega : |x-P| < a\left(\ln\frac{e\mathrm{diam}(\Omega)}{\mathrm{dist}(x,\partial\Omega)}\right)^{-\frac{n-2}{n-1}}\right\}.$$

If $n = 2$ and $p = 1$, then it follows from $N_1(\nabla u) \in L(\partial\Omega)$ that $u \in C(\bar{\Omega})$.

2). If $l < m - (n-1)/p$ then $\nabla^l u \in \mathrm{Lip}(m - l - (n-1)/p)$

All the restrictions on the set of parameters are sharp. For example, let us consider the Neumann problem for the Laplace equation. Let $\Omega \subset R^n$ be a Lipschitz domain, and let $g \in L^p(\partial\Omega)$ be a function with zero mean value. It is required to find a function u on Ω with the properties

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ \Gamma_1 - \lim \nu_P \cdot \nabla u = g(P) & \text{on } \partial\Omega, \\ N_1(\nabla u) \in L^p(\partial\Omega) & \end{cases}$$

Here $x \cdot y$ is the inner product, and ν_P is the unit outer normal at point P.

Theorem[28]. *Suppose that Ω is the unit ball in $R^n, n \geq 3$, and $1 < p < n-1$. Assume that Φ is a positive nonincreasing function with $\Phi(0+) = \infty$. Then there exists a function $g \in L^p(\partial\Omega)$ with zero mean value such that the solution of the corresponding Neumann problem for the Laplace equation does not have a limit along the domains*

$$\{x \in \Omega : |x - P| < \Phi(1 - |x|)(1 - |x|)^{1-p/(n-1)}\},$$

for a.e. $P \in \partial\Omega$. If $p = n-1$, then the same thing is true for the domains

$$\left\{x \in \Omega : |x - P| < \Phi(1 - |x|)\left(\ln \frac{e}{1 - |x|}\right)^{-\frac{n-2}{n-1}}\right\}.$$

Currently we are working on the extension of Krotov's results to the data from certain Sobolev-Besov spaces which are particularly well adapted for measuring the smoothness of data and solutions of mixed boundary value problems in Lipschitz domains.

3 Future research

3.1 Zygmund's programm in $\mathbb{R}^3$

The next natural step is to extend the mentioned in section 1.1 alternative to three dimensions. Compared with the well understood two-dimensional case, this is much more complicated. To study this case is an important matter, because one of the directions for future development of classical harmonic analysis is a study of classes of operators associated with multi-parameter groups of dilations in $\mathbb{R}^n$. In a certain sense, this is a natural "next step" after the classical Calderon-Zygmund theory and the theory of tensor products of spaces. Usually, for each such group of dilations, there is an associated maximal operator.

For instance, a classical example of a product operator is the Poisson integral for biharmonic functions on the product of two halfplanes. One can majorize the maximal Poisson operator by the strong maximal function and prove that $\mathcal{P}_y * f(x) \to f(x)$ a.e. as $|y| \to 0^+$, $y = (y_1, y_2)$, where $P_t(u)$ is the standard Poisson kernel and $\mathcal{P}_y(x) = P_{y_1}(x_1)P_{y_2}(x_2)$. The proof is based on splitting the kernel into pieces according to $|y_1| \sim 2^{-n}$, $|y_2| \sim 2^{-m}$. This is a very special case of other naturally occurring examples, such as Poisson integrals for symmetric spaces. In those spaces the chopping of the Poisson kernel in a dyadic way leads to maximal functions of special types. For example, in the simplest case of a Siegel domain with boundary space of real symmetric 2×2 matrices, the maximal function associated with the basis of three dimensional intervals with side lengths $(t, s, \sqrt{ts})$ appears.

This is why Zygmund initiated the program of determining the differentiability properties of bases associated to multi-dimensional intervals corresponding to various groups of dilations (for instance, the dilations given by $\delta_{t,s}(x, y, z) = (tx, sy, stz)$) and considered this as an extremely important topic. At the outset of this program, Zygmund considered the case of a basis $\mathcal{B}$ consisting of intervals in $\mathbb{R}^n$ whose respective side-lengths are of the form $\phi_1(x_1, \dots, x_k)$, $\dots$, $\phi_n(x_1, \dots, x_k)$, where the ϕ_i are increasing in each variable separately and assume arbitrarily small values. Noting that the k-parameter strong maximal operator differentiates $L(\log L)^{k-1}(\mathbb{R}^n)$ for any $n \geq k$, Zygmund conjectured that any such basis $\mathcal{B}$ should differentiate $L(\log L)^{k-1}(\mathbb{R}^n)$.

The first major success in this program was Córdoba's proof in [7] that the set of intervals of size $(t, s, \sqrt{ts})$ differentiates $L(\log L)(\mathbb{R}^3)$. Very recently, Fefferman and Pipher [18] have found an alternative proof of this result. Córdoba's proof provided further evidence for the Zygmund Conjecture in that he also showed that if $\mathcal{B}$ is a basis of intervals of side lengths $t, s, \phi(t, s)$ with ϕ monotonic in t and in s and taking on arbitrarily small values, then $\mathcal{B}$ differentiates $L(\log L)(\mathbb{R}^3)$.

It is now known, however, that the Zygmund Conjecture does not hold in general. In [35], Soria constructed a beautiful example of a basis of the form $(t, t\phi(s), t\psi(s))$ with non-decreasing functions $\phi(s)$ and $\psi(s)$ which has the same differentiation properties as the basis of all possible three-dimensional intervals; in particular, differentiating

$L(\log L)^2$ but not $L(\log L)$.

Soria's result sheds light on the complexity of three-dimensional case, while the two-dimensional case is now completely understood.

Based on the above results one can suggests a suitable reformulation of the Zygmund Conjecture. In particular, one might suspect the following:

Conjecture 1 *To each maximal function M associated with a translation-invariant basis of multi-dimensional intervals in $\mathbb{R}^n$ there is an integer k such that $L(\log^+ L)^k$ is the largest Orlicz class which M maps boundedly into weak L^1.*

Note that the $\mathbb{R}^2$ is proved. Recent developments also provide evidence for Conjecture 1 in higher dimensional cases. I was able to show in [36] that the differentiability properties of the strong maximal operator remain unchanged if its basis is replaced by any member of the large class of so-called *rare bases*. I am going to work on this conjecture by further development of the methodology from [39, 36]

3.2 T. Iwaniec $p^* - 1$ problem

One of the area of my future research is applications of Bellman functions to L^p-estimates for the Beurling-Ahlfors operator. A celebrated conjecture of T. Iwaniec (1982) asserts that the norm in L^p, $1 < p < \infty$, of the Beurling-Ahlfors operator is $p^* - 1$ where p^* is the maximum of p and its conjugate exponent. There are many interesting consequences of this conjecture to quasiconformal mappings and to regularity results for solutions of certain nonlinear PDE's. The Beurling-Ahlfors operator, like many other classical Caldern-Zygmund singular integrals, can be represented by certain Ito stochastic integrals. From here the powerful sharp martingale inequality techniques of D.L. Burkholder can be brought to bear to to make a progress on the conjecture. However the most significant progress in the conjecture was done recently by A. Volberg and O. Dragičević [42, 43] was based on Bellman functions techniques.

3.3 Bellman function for the Hardy-Littlewood maximal operator and Carleson embedding theorem

One can define the Bellman Function for the Hardy-Littlewood maximal function (non-centered, non-diadic) in the same was as it was defined above for the diadic one.

Let us mentioned that as is it often the case in practice, the dyadic case figures to be easier and more amenable to a Bellman function setup. The L^p-norm of the Hardy-Littlewood maximal operator $\mathcal{M}$ in one dimension was handled by Grafakos and Montgomery-Smith in [20], by a non-Bellman, essentially one-dimensional technique. It turned out that the norm of diadic maximal function M and the Hardy-Littlewood maximal function $\mathcal{M}$ different.

At the moment there are even two alternative ways to find a Bellman Function for the diadic maximal and no one for the continuous. Thus, it is a very interesting and an important problem to find a Bellman Function for the Hardy-Littlewood maximal. Additional intrigue to that problem is added by the fact, that both norms perfectly expressed in terms of the above function $\omega(t)$:

$$\|M\| = \omega(0) \qquad \|\mathcal{M}\| = \omega(-1)$$

In [30] the following Bellman function for the Carleson diadic embedding theorem was introduced

$$\tilde{\mathcal{B}}(F, f, k) = \sup \left\{ \int_K (M\varphi)^p : \ \varphi \geq 0, \ \|\varphi\|_p^p = F, \ \|\varphi\|_1 = f, \ |K| = k \right\}$$

This definition shows the relations between two Bellman functions. Thus, it is not a big surprise that Melas in the same paper[30] has found the Bellman function for the Carleson diadic embedding theorem too. It was found, again, by clever combinatorial and geometrical arguments.

Now, it natural to propose the following problem for the future research: Find Bellman PDE for the Carleson diadic embedding theorem.

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