TAUBERIAN CONDITIONS FOR GEOMETRIC MAXIMAL OPERATORS

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Abstract. Let $\mathcal{B}$ be a collection of measurable sets in $\mathbb{R}^n$. The associated geometric maximal operator $M_{\mathcal{B}}$ is defined on $L^1(\mathbb{R}^n)$ by $M_{\mathcal{B}}f(x) = \sup_{x \in R \in \mathcal{B}} \frac{1}{|R|}\int_R |f|$. If $\alpha > 0$, $M_{\mathcal{B}}$ is said to satisfy a *Tauberian condition with respect to* α if there exists a finite constant C such that for all measurable sets $E \subset \mathbb{R}^n$ the inequality $|\{x : M_{\mathcal{B}}\chi_E(x) > \alpha\}| \leq C|E|$ holds. It is shown that if $\mathcal{B}$ is a homothecy invariant collection of convex sets in $\mathbb{R}^n$ and the associated maximal operator $M_{\mathcal{B}}$ satisfies a Tauberian condition with respect to some $0 < \alpha < 1$, then $M_{\mathcal{B}}$ must satisfy a Tauberian condition with respect to γ for all $\gamma > 0$ and moreover $M_{\mathcal{B}}$ is bounded on $L^p(\mathbb{R}^n)$ for sufficiently large p. As a corollary of these results it is shown that any density basis that is a homothecy invariant collection of convex sets in $\mathbb{R}^n$ must differentiate $L^p(\mathbb{R}^n)$ for sufficiently large p.

Let $\mathcal{B}$ be a collection of measurable sets in $\mathbb{R}^n$. We define the associated geometric maximal operator $M_{\mathcal{B}}$ on $L^1(\mathbb{R}^n)$ by

$$M_{\mathcal{B}}f(x) = \sup_{x \in R \in \mathcal{B}} \frac{1}{|R|}\int_R |f| \; .$$

The operator $M_{\mathcal{B}}$ is said to satisfy a *Tauberian condition with respect to* α if there exists a finite constant C such that for any measurable set $E \subset \mathbb{R}^n$ the inequality

$$|\{x : M_{\mathcal{B}}\chi_E(x) > \alpha\}| \leq C|E|$$

holds.

This is a very weak condition on a maximal operator - weaker in fact than a restricted weak type (1,1) estimate. This is a useful condition on a maximal operator, however, as was shown by A. Córdoba and R. Fefferman in their work relating the L^p bounds of certain multiplier operators to the weak type $\left((\frac{p}{2})', (\frac{p}{2})'\right)$ bounds of associated geometric maximal operators. (See [2] for complete details.)

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Now, suppose we are given a maximal operator $M_{\mathcal{B}}$ satisfying a Tauberian condition such as, for instance,

$$\left|\left\{x : M_{\mathcal{B}}\chi_E(x) > \frac{3}{4}\right\}\right| \leq C|E| \; .$$

One might wonder whether or not $M_{\mathcal{B}}$ must be bounded on $L^p(\mathbb{R}^n)$ for $p > 1$ or whether or not $M_{\mathcal{B}}$ must satisfy any given stronger Tauberian estimate, say, $\left|\left\{x : M_{\mathcal{B}}\chi_E(x) > \frac{1}{4}\right\}\right| \leq C'|E|$. That neither of the above holds, even in the case that $\mathcal{B}$ is a homothecy invariant collection of sets, can be seen by the following example. (Recall that a collection of sets in $\mathbb{R}^n$ is said to be *homothecy invariant* if and only if any translate or dilate of any member of the collection also lies in the collection.)

Example. Let $\mathcal{B}$ be the collection of sets in $\mathbb{R}^1$ of the form $I_1 \cup I_2$, where I_1 and I_2 are intervals and $|I_2| = 2|I_1|$. Note $\mathcal{B}$ is homothecy invariant. $M_{\mathcal{B}}$ is not bounded on $L^p(\mathbb{R}^1)$ for $1 \leq p < \infty$ as $M_{\mathcal{B}}\chi_{[0,1]}(x) \geq \frac{1}{3}$ for all x in $\mathbb{R}^1$. Moreover $|\{x : M_{\mathcal{B}}\chi_{[0,1]}(x) > \frac{1}{4}\}| = \infty$, and so $M_{\mathcal{B}}$ does not satisfy a Tauberian condition with respect to $\frac{1}{4}$.

$M_{\mathcal{B}}$ does satisfy a Tauberian condition with respect to $\frac{3}{4}$, however. To see this, let E be a set of finite measure, and let $\{A_j\} \subset \mathcal{B}$ be such that $\frac{1}{|A_j|}\int_{A_j}\chi_E > \frac{3}{4}$ for each j. Now, each A_j is of the form $A_j = A_j^1 \cup A_j^2$ where A_j^1 and A_j^2 are intervals and $2|A_j^1| = |A_j^2|$. Since $\frac{1}{|A_j|}\int_{A_j}\chi_E > \frac{3}{4}$, we must have $\frac{1}{|A_j^1|}\int_{A_j^1}\chi_E > \frac{1}{4}$ and $\frac{1}{|A_j^2|}\int_{A_j^2}\chi_E > \frac{1}{4}$. So by the Vitali Covering Theorem we must have $|\cup A_j^1| \leq 12|E|$ and $|\cup A_j^2| \leq 12|E|$. So $|\cup A_j| \leq 24|E|$ and hence $|\{x : M_{\mathcal{B}}\chi_E(x) > \frac{3}{4}\}| \leq 24|E|$.

Note that in the above example the elements of $\mathcal{B}$ are not all *convex*. The primary purpose of this paper is to show that if $\mathcal{B}$ is a homothecy invariant collection of convex sets in $\mathbb{R}^n$ and the associated maximal operator $M_{\mathcal{B}}$ satisfies a Tauberian condition with respect to some $0 < \alpha < 1$, then $M_{\mathcal{B}}$ must satisfy a Tauberian condition with respect to γ for every $\gamma > 0$. As a corollary of the proof we shall see that if $\mathcal{B}$ is a homothecy invariant collection of convex sets and $M_{\mathcal{B}}$ satisfies a Tauberian condition with respect to α for some $0 < \alpha < 1$, then $M_{\mathcal{B}}$ must be bounded on $L^p(\mathbb{R}^n)$ for sufficiently large p. As a further corollary we shall see that any density basis that is a homothecy invariant collection of convex sets in $\mathbb{R}^n$ must differentiate $L^p(\mathbb{R}^n)$ for sufficiently large p.

Our proof will consist of two main parts. First we shall show the desired result in the special case that $\mathcal{B}$ is a homothecy invariant collection of rectangular parallelepipeds. Secondly we shall reduce the general case involving homothecy invariant collections of convex sets to this special case.

Proposition 1. *Let $\mathcal{B}$ be a homothecy invariant collection of rectangular parallelepipeds in $\mathbb{R}^n$. Suppose for some $0 < \gamma < 1$ there exists $0 < C_\gamma < \infty$ such that*

$$|\{x : M_{\mathcal{B}}\chi_E(x) > \gamma\}| \leq C_\gamma\,|E|$$

holds for all measurable sets E in $\mathbb{R}^n$. Then if $\alpha > 0$, there exists $0 < C_{\alpha,\gamma} < \infty$ such that

$$|\{x : M_{\mathcal{B}}\chi_E(x) > \alpha\}| \leq C_{\alpha,\gamma}\,|E|$$

holds for all measurable sets E in $\mathbb{R}^n$, where $C_{\alpha,\gamma}$ depends only on C_γ, α, γ, and the dimension n.

Proof. If $\alpha \geq \gamma$ then we may trivially set $C_{\alpha,\gamma} = C_\gamma$. So we assume without loss of generality that $0 < \alpha < \gamma$. Let E be a measurable set in $\mathbb{R}^n$. We inductively define $\mathcal{H}^k_{\mathcal{B},\gamma}(E)$ for $k = 0, 1, 2, \ldots$ by setting $\mathcal{H}^0_{\mathcal{B},\gamma}(E) = E$ and

$$\mathcal{H}^k_{\mathcal{B},\gamma}(E) = \left\{x : M_{\mathcal{B}}\chi_{\mathcal{H}^{k-1}_{\mathcal{B},\gamma}(E)}(x) \geq \gamma\right\}$$

for $k \geq 1$.

Lemma 1. *Suppose $R \in \mathcal{B}$ and $\frac{1}{|R|}\int_R \chi_E = \alpha$. Then $R \subset \mathcal{H}^{K_{\alpha,\gamma}}_{\mathcal{B},\gamma}(E)$ for some constant $K_{\alpha,\gamma}$ depending only on n, α, and γ.*

Proof. Let Q denote the unit n-cube $[0,1]^n$ in $\mathbb{R}^n$. Now, since R is a rectangular parallelepiped, there exists a linear bijection $\Lambda : \mathbb{R}^n \to \mathbb{R}^n$ such that $\{\Lambda(x) : x \in R\} = Q$.

For each set $S \in \mathbb{R}^n$ let

$$S_\Lambda = \{\Lambda(x) : x \in S\}\,.$$

Let

$$\mathcal{B}_\Lambda = \{S_\Lambda : S \in \mathcal{B}\}\,.$$

Note if U and V are measurable sets in $\mathbb{R}^N$ and $|V| \neq 0$, then $\frac{|U|}{|V|} = \frac{|U_\Lambda|}{|V_\Lambda|}$. Hence $M_{\mathcal{B}}\chi_E \geq \alpha$ on a set S in $\mathcal{B}$ if and only if $M_{\mathcal{B}_\Lambda}\chi_{E_\Lambda} \geq \alpha$ on S_Λ. Now, if $\{x : M_{\mathcal{B}}\chi_E(x) \geq \alpha\} = \cup S_j$ it follows that $\{x : M_{\mathcal{B}_\Lambda}\chi_{E_\Lambda}(x) \geq \alpha\} = \cup S_{j\Lambda}$. As $(\cup S_j)_\Lambda = \cup S_{j\Lambda}$ one sees then that

$$\left(\mathcal{H}^k_{\mathcal{B},\gamma}(E)\right)_\Lambda = \mathcal{H}^k_{\mathcal{B}_\Lambda,\gamma}(E_\Lambda)$$

holds for any positive integer k. As $R_\Lambda = Q$ we realize it suffices to prove

$$Q \subset \mathcal{H}^{K_{\alpha,\gamma}}_{\mathcal{B}_\Lambda,\gamma}(E_\Lambda)$$

for some constant $K_{\alpha,\gamma}$ depending only on n, α, and γ. As $\int_Q \chi_{E_\Lambda} > \alpha$ and $Q \in \mathcal{B}_\Lambda$ we then realize it suffices to prove the lemma in the special case that $R = Q$. Note that as $\mathcal{B}$ is homothecy invariant we may also assume without loss of generality that any n-cube in $\mathbb{R}^n$ with sides parallel to the axes lies in $\mathcal{B}$.

So, we now suppose without loss of generality that $R = Q$, all n-cubes in $\mathbb{R}^n$ whose sides are parallel to the axes lie in $\mathcal{B}$, and $\int_Q \chi_E = \alpha$. We take the Calderón-Zygmund decomposition of $\chi_{E\cap Q}$ with respect to γ yielding a collection of cubes $\{Q_j\}$ in Q with sides parallel to the axes. In particular the collection of cubes $\{Q_j\}$ is such that $\frac{1}{|Q_j|}\int_{Q_j} \chi_E > \gamma$ for each j and $E \cap Q \subset \cup Q_j$ almost everywhere. Note none of the cubes Q_j is Q itself as $\frac{1}{|Q|}\int_Q \chi_E = \alpha < \gamma$. Also note that each Q_j is a dyadic cube and hence has a unique parent dyadic cube. For any constant $c > 1$, we let cQ_j denote the cube containing Q_j that has sidelength c times that of Q_j and also has a common corner with Q_j and the parent cube of Q_j.

Let now $E_0 = E \cap Q$, $E_1 = \cup Q_j$, and, for $k \geq 2$,

$$E_k = \bigcup_j \left(\frac{1}{\gamma}\right)^{(k-1)/n} Q_j \ .$$

Note that since

$$\frac{\left|(\frac{1}{\gamma})^{\frac{k}{n}} Q_j\right|}{\left|(\frac{1}{\gamma})^{\frac{k+1}{n}} Q_j\right|} = \gamma$$

we have $M_{\mathcal{B}}\chi_{E_k} \geq \gamma$ on E_{k+1}. Also observe that since the average of χ_E over each Q_k exceeds γ we have $E_1 \subset \mathcal{H}^1_{\mathcal{B},\gamma}(E)$, and as $M_{\mathcal{B}}\chi_{E_k} \geq \gamma$ on E_{k+1} we have $E_k \subset \mathcal{H}^k_{\mathcal{B},\gamma}(E)$ for each k.

Let now N be a positive integer such that $\left(\frac{1}{\gamma}\right)^N \geq \gamma \cdot 2^n$. Let Q_j^* denote the parent cube of Q_j. Now, since

$$\frac{\left|\left(\frac{1}{\gamma}\right)^{\frac{N}{n}} Q_j\right|}{\left|Q_j^*\right|} \geq \left(\frac{1}{\gamma}\right)^N \cdot \frac{1}{2^n} \geq \gamma$$

we have

$$\frac{\left|E_{N+1} \cap Q_j^*\right|}{\left|Q_j^*\right|} \geq \gamma,$$

and so $M_{\mathcal{B}}\chi_{E_{N+1}} \geq \gamma$ on Q_j^*.

Let now $Q_{j_1}, Q_{j_2}, \ldots$ be elements of $\{Q_j\}$ such that the $Q_{j_i}^*$ have disjoint interiors and such that $\left|\cup Q_{j_i}^*\right| = |\cup Q_k^*|$. Note each $Q_{j_i}^*$ is contained in Q since $Q \notin \{Q_i\}$.

Fix k. We are now going to show $\left|E \cap Q_{j_k}^*\right| \leq \gamma' \left|Q_{j_k}^*\right|$ for some constant $\gamma' = \gamma'(\gamma)$ less than 1 and subsequently that $\left|Q \cap \mathcal{H}^{N+2}_{\mathcal{B},\gamma}(E)\right| \geq \frac{1}{\gamma'}|E_0|$.

Let $\tilde{\gamma} = \gamma + \frac{1-\gamma}{2} = \frac{\gamma+1}{2}$. Note since $0 < \gamma < 1$ we have $0 < \gamma < \tilde{\gamma} < 1$.

Now, we say a cube Q_j is of *Type I* if $\frac{1}{|Q_j|}\int_{Q_j}\chi_E \geq \tilde{\gamma}$ and of *Type II* otherwise. We define the sets $\mathbf{U}_1$, $\mathbf{U}_2$, and $\mathbf{U}$ by

$$\mathbf{U}_1 = \bigcup_{\substack{j: Q_j \subset Q^*_{j_k} \\ Q_j of Type I}} Q_j \ , \quad \mathbf{U}_2 = \bigcup_{\substack{j: Q_j \subset Q^*_{j_k} \\ Q_j of Type II}} Q_j \ , \quad \text{and} \quad \mathbf{U} = \bigcup_{\substack{j: Q_j \subset Q^*_{j_k} \\ Q_j of Type II}} Q_j \cap E \ .$$

It is clear that

$$|\mathbf{U}_2| \leq |Q^*_{j_k}| - |\mathbf{U}_1|.$$

Observe that

$$|\mathbf{U}_1| \leq \frac{\gamma}{\tilde{\gamma}} \left|Q^*_{j_k}\right|$$

as otherwise

$$\frac{1}{\left|Q^*_{j_k}\right|}\int_{Q^*_{j_k}} \chi_E > \tilde{\gamma}\cdot\frac{\gamma}{\tilde{\gamma}} = \gamma,$$

contradicting the fact that $Q^*_{j_k}$ was *not* one of the selected cubes $\{Q_j\}$. Now,

$$\begin{aligned}
|E \cap Q^*_{j_k}| &= |E \cap \mathbf{U}_1| + |E \cap \mathbf{U}_2| \\
&\leq |\mathbf{U}_1| + |\mathbf{U}| \\
&\leq |\mathbf{U}_1| + \tilde{\gamma}\,|\mathbf{U}_2| \\
&\leq |\mathbf{U}_1| + \tilde{\gamma}\left(\left|Q^*_{j_k}\right| - |\mathbf{U}_1|\right) \\
&= \tilde{\gamma}\left|Q^*_{j_k}\right| + (1-\tilde{\gamma})\,|\mathbf{U}_1| \\
&\leq \tilde{\gamma}\left|Q^*_{j_k}\right| + \frac{\gamma}{\tilde{\gamma}}(1-\tilde{\gamma})\left|Q^*_{j_k}\right| \\
&= \left|Q^*_{j_k}\right|\left(\frac{\gamma}{\tilde{\gamma}} + \tilde{\gamma}\left(1 - \frac{\gamma}{\tilde{\gamma}}\right)\right).
\end{aligned}$$

Set

$$\gamma' = \frac{\gamma}{\tilde{\gamma}} + \tilde{\gamma}\left(1 - \frac{\gamma}{\tilde{\gamma}}\right).$$

Note since $0 < \gamma < \tilde{\gamma} < 1$ we have $0 < \gamma' < 1$. Moreover

$$\begin{aligned}
\left|\left\{x \in Q : M_{\mathcal{B}}\chi_{E_{N+1}}(x) \geq \gamma\right\}\right| &\geq \left|\cup Q^*_j\right| \\
&= \sum \left|Q^*_{j_k}\right| \\
&\geq \frac{1}{\gamma'}\sum\left|E \cap Q^*_{j_k}\right| \\
&\geq \frac{1}{\gamma'}\,|E_0|\,.
\end{aligned}$$

In particular,

$$\left|Q \cap \mathcal{H}^{N+2}_{\mathcal{B},\gamma}(E)\right| \geq \frac{1}{\gamma'}\,|E_0|\,.$$

Note that if $\left|Q \cap \mathcal{H}_{\mathcal{B},\gamma}^{N+2}(E)\right| \geq \gamma$ we have $Q \subset \mathcal{H}_{\mathcal{B},\gamma}^{(N+2)+1}(E)$. Otherwise by the above argument we may obtain

$$\begin{aligned}\left|Q \cap \mathcal{H}_{\mathcal{B},\gamma}^{2(N+2)}(E)\right| &\geq \frac{1}{\gamma'}\left|H_{\mathcal{B},\gamma}^{N+2}(E) \cap Q\right| \\ &\geq \left(\frac{1}{\gamma'}\right)^2 |E_0|.\end{aligned}$$

More generally, if $\left|Q \cap \mathcal{H}_{\mathcal{B},\gamma}^{j(N+2)}(E)\right| \geq \gamma$ we have $Q \subset \mathcal{H}_{\mathcal{B},\gamma}^{j(N+2)+1}(E)$, or otherwise we may obtain

$$\left|Q \cap \mathcal{H}_{\mathcal{B},\gamma}^{(j+1)(N+2)}(E)\right| \geq \left(\frac{1}{\gamma'}\right)^{j+1} |E_0|.$$

Now, let $\tilde{N}$ be a positive integer such that $\alpha \cdot \left(\frac{1}{\gamma'}\right)^{\tilde{N}} \geq \gamma$. As $|E_0| = \alpha$ we have $\left(\frac{1}{\gamma'}\right)^{\tilde{N}} |E_0| \geq \gamma$. Hence for some $m \leq (N+2)\cdot\tilde{N}$ we have $\left|Q \cap \mathcal{H}_{\mathcal{B},\gamma}^{m}(E)\right| \geq \gamma$. In particular, $Q \subset \mathcal{H}_{\mathcal{B},\gamma}^{(N+2)\tilde{N}+1}(E)$. As any integer greater than or equal to $\frac{\log^+(\gamma\cdot 2^n)}{\log(\frac{1}{\gamma})}$ would be acceptable for N and any integer greater than or equal to $\frac{-\log(\frac{\gamma}{\alpha})}{\log(\frac{2\gamma}{\gamma+1}+\frac{1-\gamma}{2})}$ would be acceptable for $\tilde{N}$, we obtain the lemma, where

$$K_{\alpha,\gamma} = \left\lceil \frac{-\log(\frac{\gamma}{\alpha})}{\log(\frac{2\gamma}{\gamma+1}+\frac{1-\gamma}{2})} \right\rceil \cdot \left\lceil 2 + \frac{\log^+(\gamma\cdot 2^n)}{\log(\frac{1}{\gamma})} \right\rceil + 1. \tag{1}$$

□

We now complete the proof of Proposition 1. As $|\{x : M_{\mathcal{B}}\chi_E(x) > \gamma\}| \leq C\,|E|$ for every measurable set E if and only if $|\{x : M_{\mathcal{B}}\chi_E(x) \geq \gamma\}| \leq C\,|E|$ for every measurable set E, by the Tauberian condition on $M_{\mathcal{B}}$ we have

$$\left|\mathcal{H}_{\mathcal{B},\gamma}^{k+1}(E)\right| \leq C_\gamma \left|\mathcal{H}_{\mathcal{B},\gamma}^{k}(E)\right|$$

holds for any positive integer k and any measurable set E. An immediate consequence of the above lemma is that $\{x : M_{\mathcal{B}}\chi_E(x) > \alpha\} \subset \mathcal{H}_{\mathcal{B},\gamma}^{K_{\alpha,\gamma}}(E)$, and hence

$$\begin{aligned}|\{x : M_{\mathcal{B}}\chi_E(x) > \alpha\}| &\leq \left|\mathcal{H}_{\mathcal{B},\gamma}^{K_{\alpha,\gamma}}(E)\right| \\ &\leq C_\gamma \left|\mathcal{H}_{\mathcal{B},\gamma}^{K_{\alpha,\gamma}-1}(E)\right| \\ &\leq \ldots \leq C_\gamma^{K_{\alpha,\gamma}}|E|.\end{aligned}$$

So $|\{x : M_{\mathcal{B}}\chi_E(x) > \alpha\}| \leq C_{\alpha,\gamma}\,|E|$, where $C_{\alpha,\gamma} = C_\gamma^{K_{\alpha,\gamma}}$ and $K_{\alpha,\gamma}$ is as in (1).

□

In Proposition 1 $\mathcal{B}$ is a homothecy invariant collection of rectangular parallelepipeds. The following theorem is a generalization of Proposition 1 in that we allow $\mathcal{B}$ to be a homothecy invariant collection of convex sets.

Theorem 1. *Let $\mathcal{B}$ be a homothecy invariant collection of convex sets in $\mathbb{R}^n$. Suppose for some $0 < \alpha < 1$ there exists $0 < C_\alpha < \infty$ such that*

$$|\{x : M_{\mathcal{B}}\chi_E(x) > \alpha\}| \leq C_\alpha |E|$$

holds for all measurable sets E in $\mathbb{R}^n$. Then if $\delta > 0$, there exists $0 < C_{\alpha,\delta} < \infty$ such that

$$|\{x : M_{\mathcal{B}}\chi_E(x) > \delta\}| \leq C_{\alpha,\delta} |E|$$

holds for all measurable sets E in $\mathbb{R}^n$, where $C_{\alpha,\delta}$ depends only on C_α, α, δ, and the dimension n.

Proof. Given an ellipsoid $\mathcal{E}$ in $\mathbb{R}^n$ and $c > 0$, we let $c\mathcal{E}$ denote the c-fold dilate of $\mathcal{E}$ that has the same center and orientation as $\mathcal{E}$.

Let $S \in \mathcal{B}$. As was proven by F. John in [4] (see also the related article [1] by K. Ball), since S is convex there exists an ellipsoid $\mathcal{E}_S$ contained in S such that $S \subset n\mathcal{E}_S$. Let R_S be a rectangular parallelepiped containing $n\mathcal{E}_S$ of smallest possible volume. Note $|R_S| < 2^n |n\mathcal{E}_S|$ and hence $|R_S| < 2^n \cdot n^n |S|$. Moreover, letting cS denote the c-fold dilate of S about the center of $\mathcal{E}_S$ we have $R_S \subset 2nS$ since $R_S \subset 2n\mathcal{E}_S$ and $2n\mathcal{E}_S \subset 2nS$.

Let $\tilde{\mathcal{B}} = \{R_S : S \in \mathcal{B}\}$. We may assume without loss of generality that the $\mathcal{E}_S$ and R_S above are such that $\tilde{\mathcal{B}}$ is homothecy invariant.

Note that $M_{\tilde{\mathcal{B}}} f(x) \leq 2^n \cdot n^n M_{\mathcal{B}} f(x)$.

We now fix γ such that $0 < \alpha < \gamma < 1$.

Let $\rho = \frac{1}{2^n \cdot n^n}$. Also let

$$\epsilon = \frac{\gamma - \alpha}{2 - \gamma - \alpha}\, \rho \quad \text{and} \quad N = \left\lceil \frac{\log\left(1 - \frac{2(1-\gamma)}{2-\gamma-\alpha}\right)}{\log\left(1 - \rho - \frac{\gamma-\alpha}{2-\gamma-\alpha}\, \rho\right)} \right\rceil . \tag{2}$$

One can show that

$$\rho\, \frac{1 - (1 - \rho - \epsilon)^{N+1}}{\rho + \epsilon} > \frac{1 - \gamma}{1 - \alpha} .$$

We will need the following technical lemma.

Lemma 2. *Let $\epsilon > 0$ be as above and S be a convex set in $Q = [0,1]^n$. Let $m \in \mathbb{N}$ be the unique positive integer such that*

$$\frac{\epsilon}{4n} \leq \sqrt{n} 2^{-m} < \frac{\epsilon}{2n} .$$

Then there exists a set of cubes $\{Q_j\}$ of sidelength 2^{-m} such that *i) all the cubes Q_j lie in Q and are members of the mesh $\mathcal{M}_m$ of dyadic cubes of sidelength 2^{-m},*

ii) each Q_i is disjoint from S, and

iii) $|\cup Q_i \cup S| \geq 1 - \epsilon$.

Proof. Let $\mathcal{C}$ be the set of cubes in the mesh $\mathcal{M}_m$ that lie in Q and are disjoint from S. Suppose $x \in Q \setminus S$ and $d(x, S) > \frac{\epsilon}{2n}$. Then as the diameter of any cube in $\mathcal{M}_m$ is less than $\frac{\epsilon}{2n}$ we have $x \in Q_j$ for some Q_j in $\mathcal{C}$. So

$$\left\{x \in Q : d(x,S) > \frac{\epsilon}{2n}\right\} \subset \bigcup_{Q_j \in \mathcal{C}} Q_j .$$

Now, since S is convex,

$$\left|\left\{x \in Q : 0 < d(x,S) < \frac{\epsilon}{2n}\right\}\right| < 2n \cdot \frac{\epsilon}{2n} = \epsilon \ ,$$

so the desired result holds. □

If S is a set in $\mathbb{R}^n$ and τ is a translation operator given by $\tau f(x) = f(x-\sigma)$ for some $\sigma \in \mathbb{R}^n$, we let τS denote the set such that $\chi_{\tau S}(x) = \chi_S(x - \sigma)$. For each $c > 0$ and set S in $\mathbb{R}^n$ we define the set $\delta_c S$ to be such that $\chi_{\delta_c S}(x) = \chi_S\left(\frac{1}{c}x\right)$.

Lemma 3. *Suppose $R \in \tilde{\mathcal{B}}$. Let $S \in \mathcal{B}$ such that $S \subset R$, $|R| < 2^n \cdot n^n |S|$, and $R \subset 2nS$. Then there exists an a.e. disjoint collection $\{S_j\}$ of translates of dilates of S and a collection of translation operators $\{\tau_j\}$ such that $S_j \subset R$ for each j, $|\cup S_j| > \frac{1-\gamma}{1-\alpha}|R|$, and $R \subset \tau_j \delta_{2^{Nm+n}} S_j$ for each j. (Here m is as is given by Lemma 2 and N is as in (2).)*

Proof. As the techniques of this proof are invariant under affine transformation, we may assume without loss of generality that $R = Q = [0,1]^n$.

Note $\frac{|S|}{|R|} > \rho$.

By Lemma 2, there exist a collection $\{Q_j\}$ of (a.e.) disjoint n-cubes contained in R and disjoint from S lying in the mesh $\mathcal{M}_m$ such that $|\cup Q_i \cup S| \geq 1 - \epsilon$.

Let now $\{\tau_j\}$ be a collection of translation operators such that $Q_j = \tau_j \delta_{2^{-m}} R$ for each j.

Let $S_{1,j} = \tau_j \delta_{2^{-m}} S$. Note

$$|S \cup (\cup S_{1,j})| \geq \rho + (1 - \rho - \epsilon)\,\rho$$

since $|(\cup Q_j) \cup S| \geq 1 - \epsilon$ and $|S| > \rho$.

Let $S_1 = S \cup (\cup S_{1,j})$. Let $S_{2,j} = \tau_j \delta_{2^{-m}} S_1$. Observe that

$$|S \cup (\cup S_{2,j})| \geq \rho + (1 - \rho - \epsilon)\,\rho + (1 - \rho - \epsilon)^2\,\rho \ .$$

Let $S_2 = S \cup (\cup S_{2,j})$.

We proceed by induction. $S_{k+1,j}$ and S_{k+1} may be obtained from S_k via

$$S_{k+1,j} = \tau_j \delta_{2^{-m}} S_k$$

and

$$S_{k+1} = S \cup \left(\cup S_{k+1,j}\right) .$$

Note

$$\left|S \cup \left(\cup_j S_{k+1,j}\right)\right| \geq \rho + (1-\rho-\epsilon)\,\rho + \ldots + (1-\rho-\epsilon)^{k+1}\,\rho.$$

Recall now N is such that

$$\rho\,\frac{1-(1-\rho-\epsilon)^{N+1}}{\rho+\epsilon} > \frac{1-\gamma}{1-\alpha} .$$

So

$$\begin{aligned} |S_N| &\geq \rho + (1-\rho-\epsilon)\,\rho + \ldots + (1-\rho-\epsilon)^{N}\,\rho \\ &= \rho\,\frac{1-(1-\rho-\epsilon)^{N+1}}{1-(1-\rho-\epsilon)} = \rho\,\frac{1-(1-\rho-\epsilon)^{N+1}}{\rho+\epsilon} \\ &> \frac{1-\gamma}{1-\alpha}. \end{aligned}$$

Note also there exists a collection of translation operators $\tau_{j,k}$ such that

$$S_N = S \cup \left(\cup_{j=1}^{N} \cup_k \tau_{j,k}\delta_{2^{-jm}} S\right) ,$$

where the union above is disjoint. So in particular S_N may be expressed as the disjoint union $\cup S'_j$, where $\left|\cup S'_j\right| > \frac{1-\gamma}{1-\alpha}$ and each S'_j is a translate of a dilate of S such that $S'_j \subset R$. Moreover there exists a set of translation operators $\left\{\tau'_j\right\}$ such that $S \subset \tau'_j \delta_{2^{Nm}} S'_j$ for each j. Since $R \subset 2nS$, there also exists a collection of translation operators $\left\{\tau''_j\right\}$ such that $R \subset \tau''_j \delta_{2^{Nm+n}} S'_j$ for each j. Relabeling $\left\{S'_j\right\}$ as $\{S_j\}$ and $\left\{\tau''_j\right\}$ as $\{\tau_j\}$, we complete the proof of the lemma. □

The following lemma shows that, since $M_{\mathcal{B}}$ satisfies a Tauberian condition with respect to α, the maximal operator $M_{\tilde{\mathcal{B}}}$ satisfies a Tauberian condition with respect to any γ greater than α.

Lemma 4. *If $\alpha < \gamma < 1$, there exists $0 < C'_{\alpha,\gamma} < \infty$ such that*

$$\left|\{x : M_{\tilde{\mathcal{B}}}\chi_E(x) > \gamma\}\right| \leq C'_{\alpha,\gamma}\,|E|$$

holds for all measurable sets E in $\mathbb{R}^n$, where $C'_{\alpha,\gamma}$ depends only on C_α, α, γ, and the dimension n.

Proof. Let E be a measurable set in $\mathbb{R}^n$. Suppose $R \in \tilde{\mathcal{B}}$ and $\frac{1}{|R|}\int_R \chi_E > \gamma$. Let $\{S_j\}$ be as in Lemma 3. Then there exists $\tilde{S} \in \{S_j\}$ such that $\frac{1}{|\tilde{S}|}\int_{\tilde{S}} \chi_E > \alpha$, as otherwise

$$\begin{aligned}|E \cap R| &\leq \left(\frac{1-\gamma}{1-\alpha}\cdot \alpha + 1 \cdot \left(1 - \frac{1-\gamma}{1-\alpha}\right)\right)|R| \\ &= \gamma\,|R|\ ,\end{aligned}$$

contradicting that $|E \cap R|/|R| > \gamma$. By Lemma 3 we have $R \subset \tau\delta_{2^{Nm+n}}\tilde{S}$ for some translation operator τ. We now define $\Delta_{\alpha,\gamma}$ by

$$\Delta_{\alpha,\gamma} = 1 + \frac{n\log 2}{\log(\frac{1}{\alpha})}\left(\left\lceil \frac{\log\left(1 - \frac{2(1-\gamma)}{2-\gamma-\alpha}\right)}{\log\left(1 - \frac{1}{2^n n^n} - \frac{\gamma-\alpha}{2-\gamma-\alpha}\frac{1}{2^n n^n}\right)}\right\rceil \left\lceil - \frac{\log\left(\frac{\gamma-\alpha}{2-\gamma-\alpha}\frac{1}{2^{n+1}n^{n+\frac{3}{2}}}\right)}{\log 2}\right\rceil + n\right) .$$

One can show that $\Delta_{\alpha,\gamma}$ satisfies the inequality

$$\left(\frac{1}{\alpha}\right)^{\frac{1}{n}(\Delta_{\alpha,\gamma}-1)} \geq 2^{Nm+n}.$$

Note then $R \subset \mathcal{H}_{\mathcal{B},\alpha}^{\Delta_{\alpha,\gamma}-1}(\tilde{S})$ and in particular $R \subset \mathcal{H}_{\mathcal{B},\alpha}^{\Delta_{\alpha,\gamma}}(E)$. As R was arbitrary in $\tilde{\mathcal{B}}$ subject to the condition that $\frac{1}{|R|}\int_R \chi_E > \gamma$ we then have $\{x : M_{\tilde{\mathcal{B}}}\chi_E(x) > \gamma\} \subset \mathcal{H}_{\mathcal{B},\alpha}^{\Delta_{\alpha,\gamma}}(E)$. By the Tauberian condition on $M_{\mathcal{B}}$ we then have that $|\{x : M_{\tilde{\mathcal{B}}}\chi_E(x) > \gamma\}| \leq C_\alpha^{\Delta_{\alpha,\gamma}}|E|$. As $C_\alpha^{\Delta_{\alpha,\gamma}}$ depends only on C_α, α, γ, and n, the desired result holds. □

We now come to the end of the proof of the main theorem. We may assume $0 < \delta < \alpha$ without loss of generality. The hypotheses of the theorem and Lemma 4 and its proof imply that $|\{x : M_{\tilde{\mathcal{B}}}\chi_E(x) > \gamma\}| \leq C_\alpha^{\Delta_{\alpha,\gamma}}|E|$ for $\alpha < \gamma < 1$. We now set $\gamma = \tilde{\alpha} = \frac{1+\alpha}{2}$. Since $\tilde{\mathcal{B}}$ is a homothecy invariant collection of rectangular parallelepipeds, by the closing comments of the proof of Proposition 1 we have that for any measurable set E in $\mathbb{R}^n$

$$\left|\left\{x : M_{\tilde{\mathcal{B}}}\chi_E(x) > \frac{\delta}{2^n n^n}\right\}\right| \leq C_\alpha^{\Delta_{\alpha,\tilde{\alpha}}K_{\frac{\delta}{2^n n^n},\tilde{\alpha}}}|E|\ .$$

Since $M_{\mathcal{B}}f(x) \leq 2^n n^n M_{\tilde{\mathcal{B}}}f(x)$ we then have

$$|\{x : M_{\mathcal{B}}\chi_E(x) > \delta\}| \leq C_\alpha^{\Delta_{\alpha,\tilde{\alpha}}K_{\frac{\delta}{2^n n^n},\tilde{\alpha}}}|E|\ .$$

As $\Delta_{\alpha,\tilde{\alpha}}$ and $K_{\frac{\delta}{2^n n^n},\tilde{\alpha}}$ depend only on α, δ, and n, the desired result holds. □

We now show that the proof of the above result implies that, if $\mathcal{B}$ is a homothecy invariant collection of convex sets in $\mathbb{R}^n$ and the associated maximal operator $M_{\mathcal{B}}$ satisfies a Tauberian condition with respect to some $0 < \alpha < 1$, then $M_{\mathcal{B}}$ must be bounded on $L^p(\mathbb{R}^n)$ for sufficiently large p.

Corollary 1. *Let $\mathcal{B}$ be a homothecy invariant collection of convex sets in $\mathbb{R}^n$. Suppose for some $0 < \alpha < 1$ there exists a positive finite constant C_α such that*

$$|\{x : M_{\mathcal{B}}\chi_E(x) > \alpha\}| \leq C_\alpha \, |E|$$

holds for every measurable set E in $\mathbb{R}^n$. Then $M_{\mathcal{B}}$ is bounded on $L^p(\mathbb{R}^n)$ for sufficiently large p. In particular, there exists $p_\alpha < \infty$ depending only on α, n, and C_α such that $M_{\mathcal{B}}$ is bounded on $L^p\left(\mathbb{R}^n\right)$ for all $p > p_\alpha$.

Proof. Let $\delta < \min(\frac{1}{100}, \alpha)$. By the closing remarks of the proof of Theorem 1 we have that

$$\begin{aligned}
|\{x : M_{\mathcal{B}}\chi_E(x) > \delta\}| &\leq C_\alpha^{\Delta_{\alpha,\tilde{\alpha}} K_{\frac{\delta}{2^n n^n}, \tilde{\alpha}}} \, |E| \\
&\leq C_\alpha^{\Delta_{\alpha,\tilde{\alpha}}\left(\left\lceil \frac{-\log(\frac{\tilde{\alpha}2^n n^n}{\delta})}{\log(\frac{2\tilde{\alpha}}{\tilde{\alpha}+1}+\frac{1-\tilde{\alpha}}{2})}\right\rceil \cdot \left\lceil 2+\frac{\log^+(2^n\tilde{\alpha})}{\log(\frac{1}{\tilde{\alpha}})}\right\rceil + 1\right)} \, |E| \\
&\leq C_\alpha^{\Delta_{\alpha,\tilde{\alpha}}} C_\alpha^{2\Delta_{\alpha,\tilde{\alpha}} \frac{-\log(\frac{\tilde{\alpha}2^n n^n}{\delta})}{\log(\frac{2\tilde{\alpha}}{\tilde{\alpha}+1}+\frac{1-\tilde{\alpha}}{2})} \cdot \left\lceil 2+\frac{\log^+(2^n\tilde{\alpha})}{\log(\frac{1}{\tilde{\alpha}})}\right\rceil} \, |E| \\
&\leq C_\alpha^{\Delta_{\alpha,\tilde{\alpha}}} \left(\frac{\tilde{\alpha}\cdot 2^n \cdot n^n}{\delta}\right)^{\frac{-2\log C_\alpha \left\lceil 2+\frac{\log^+(2^n\tilde{\alpha})}{\log(\frac{1}{\tilde{\alpha}})}\right\rceil \Delta_{\alpha,\tilde{\alpha}}}{\log\left(\frac{2\tilde{\alpha}}{\tilde{\alpha}+1}+\frac{1-\tilde{\alpha}}{2}\right)}} \, |E| \, .
\end{aligned}$$

Hence $M_{\mathcal{B}}$ is of restricted weak type (p_α, p_α), where

$$p_\alpha = \frac{-2\log C_\alpha \left\lceil 2 + \frac{\log^+(2^n\tilde{\alpha})}{\log(\frac{1}{\tilde{\alpha}})}\right\rceil \Delta_{\alpha,\tilde{\alpha}}}{\log\left(\frac{2\tilde{\alpha}}{\tilde{\alpha}+1} + \frac{1-\tilde{\alpha}}{2}\right)} \, ,$$

and hence $M_{\mathcal{B}}$ is bounded on $L^p\left(\mathbb{R}^n\right)$ for any $p > p_\alpha$. As p_α depends only on α, n, and C_α, the desired result follows. ☐

Recall that a collection of sets in $\mathbb{R}^n$ is said to be a *density basis* if it differentiates $L^\infty(\mathbb{R}^n)$. We conclude this paper by observing the rather striking result that any density basis consisting of a homothecy invariant collection of convex sets in $\mathbb{R}^n$ must differentiate $L^p\left(\mathbb{R}^n\right)$ for sufficiently large p.

Corollary 2. *Let $\mathcal{B}$ be a density basis that is a homothecy invariant collection of convex sets in $\mathbb{R}^n$. Then $\mathcal{B}$ differentiates $L^p\left(\mathbb{R}^n\right)$ for sufficiently large p.*

Proof. Suppose $\mathcal{B}$ is a density basis that is a homothecy invariant collection of convex sets in $\mathbb{R}^n$. Then since $\mathcal{B}$ is a Busemann-Feller basis that is invariant by homothecies, we know for some $0 < C < \infty$ that $\left|\left\{x : M_{\mathcal{B}}\chi_E(x) > \frac{1}{2}\right\}\right| \leq C\,|E|$ holds for all measurable sets E in $\mathbb{R}^n$. (See p. 69 of [3] for a proof of this result.) By Corollary 1 we then have that

$M_{\mathcal{B}}$ is bounded on $L^p(\mathbb{R}^n)$ for sufficiently large p and hence $\mathcal{B}$ differentiates $L^p(\mathbb{R}^n)$ for sufficiently large p.

□

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