PROPERTIES OF THE MAXIMAL OPERATORS ASSOCIATED TO BASES OF RECTANGLES IN $\mathbb{R}^3$

ALEXANDER STOKOLOS

Abstract. This paper is an attempt to understand a phenomena of maximal operators associated to bases of three dimensional rectangles of dimensions $(t, 1/t, s)$ within a framework of more general Soria bases. Jessen-Marcinkiewicz-Zygmund Theorem implies that the maximal operator associated with a Soria basis continuously maps $L\log^2 L$ into $L^{1,\infty}$. We give a simple geometric condition which guarantees that $L\log^2 L$ class cannot be enlarged. The proof of the result is a further development of the methods from [8] and is related to the theorems of A.Córdoba [1], F.Soria [7], R. Fefferman and J. Pipher [4].

In this paper we will deal with translation invariant collections consisting of certain multi-dimensional rectangles (i.e. Cartesian products of one-dimensional intervals). We will call them bases. The main result about a basis consisting of the all possible n-dimensional rectangles is the famous Jessen-Marcinkiewicz-Zygmund Theorem. The quantitative version of this theorem is this weak type $L(\log^+ L)^{n-1}$ estimate for the strong maximal operator

$$M_s f(x) = \sup_{\text{all } R} \frac{1}{|R|} \int_{R \ni x} |f(y)| dy :$$

$$|\{M_s f > \lambda\}| \le \int \frac{|f|}{\lambda} \left(1 + \log^+ \frac{|f|}{\lambda}\right)^{n-1}. \tag{1}$$

We will say that a basis $\mathcal{B}$ is weak type $L(\log^+ L)^j$ if condition (1) holds with $n-1$ replaced by j and $M_s f$ is replaced by

$$M_{\mathcal{B}} f(x) = \sup_{R \in \mathcal{B}} \frac{1}{|R|} \int_{R \ni x} |f(y)| dy.$$

The estimate (1) was established by N. Fava [3] and independently by M. de Guzmán [5]. Testing the inequality on the characteristic function of a unit disc indicates that it is sharp. One can look on it as a result of action of the corresponding maximal operator on δ function (for more discussions see "final remarks" on p. 3240 of [4]).

This estimate gets worse by a logarithmic factor with each increment of the dimension. On the other hand, A. Zygmund [9] demonstrated that the basis consisting of the Cartesian product of k-dimensional cubes and $n-k$ one-dimensional intervals has a weak type $L(\log^+ L)^{n-k}$. Thus, it would be natural to expect that

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the properties of the bases become worse with the addition of extra degree of freedom and vice versa, the properties of the bases become better with the reduction of the degree of freedom. A. Zygmund has conjectured that if the basis consists of the n-dimensional rectangles which have side length as functions of the same k independent variables ($k < n$), then the basis should behave like the basis of all k-dimensional rectangles. For example, a basis of two dimensional rectangles of side-length $(t, \phi(t))$ with nondecreasing $\phi(t)$ has a weak type (1,1). A. Córdoba [1] proved Zygmund's conjecture for an important particular case, establishing that the basis of three dimensional rectangles of dimensions $(t, s, h(t, s))$ continuously maps $L \log^+ L$ into weak L when $h(t, s)$ is a function non-decreasing in each variable.

In spite of the progress that has been made towards the proof of the hypothesis, it turns out that in such generality the conjecture is false. F. Soria [7] has constructed a beautiful example of the basis of three dimensional rectangles of dimensions $(t, t\phi(s), t\psi(s))$ with the non-decreasing functions $\phi(s)$ and $\psi(s)$, which has the same property as the basis of all possible three dimensional rectangles.

Soria's result shed a light on the complexity of the three dimensional case, while the two dimensional case now is totally understood.

Let us be more specific. We call two rectangles R and R' comparable and denote this $R \sim R'$ if there exists a translation placing one of them inside the other one. In the opposite case we call them *incomparable* and write $R \not\sim R'$.

Now, for every rectangle $R \in \mathcal{B}$ we denote by R^* the concentric rectangle of minimal measure containing R with side lengths of the form 2^k, $k \in \mathbb{Z}$. Thus, to every basis $\mathcal{B}$ we attach, in a natural way, another basis $\mathcal{B}^* = \{R^* \mid R \in \mathcal{B}\}$ - a dyadic skeleton of the basis $\mathcal{B}$. It is clear that $\mathcal{B}$ and $\mathcal{B}^*$ have the same weak type estimates.

In [8] the following property

$$\exists k > 1 \; \forall R_1, \ldots, R_k \in \mathcal{B}^* \; \exists i \neq j, \qquad R_i \sim R_j, \tag{w}$$

and an alternative one

$$\forall k > 1 \; \exists R_1, \ldots, R_k \in \mathcal{B}^* \; \forall i \neq j, \qquad R_i \not\sim R_j. \tag{s}$$

were introduced.

It turns out that in the two dimensional case the s-property makes a basis bad. Any basis possessing the s-property behaves like the basis of all two dimensional intervals while any basis possessing the w-property behaves like the basis of all two dimensional cubes (for more details see [8]).

A model two dimensional basis with the s-property is provided by the basis of rectangles of dimensions $(t, 1/t)$. It is a weak type $L \log^+ L$ basis and this is the best possible estimate. The appearance of the logarithmic factor is a result of the integration of the $1/t$ function.

Switching to three dimensional space and considering the basis of three dimensional rectangles of dimensions $(t, 1/t, s)$ one can expect that the extra degree of freedom brings an extra logarithm into the weak type estimate. Surprisingly, this is not the case. F. Soria [7] noticed that this basis is a weak type $L \log^+ L$

basis too. And this happened again because any two different rectangles from $(t, 1/t)$-basis are incomparable! Thus, the three dimensional case turns out to be totally different from the two dimensional case - *both comparableness and incomparableness can improve the properties of bases.*

The problem of investigating the three dimensional case is in the lack of covering methodology. The arsenal of tools is very limited: there are only standard Vitaly (see e.g. [6]), Cordoba-Fefferman [2], Cordoba [1] and Fefferman-Pipher [4] covering arguments. Further development of the general case seems quite difficult. Thus, it would be natural to study some particular important bases. For instance, a basis consisting of Cartesian products of the two dimensional rectangles forming a certain basis B in the XY-plane (we will call it a *projection basis*) and arbitrary one-dimensional intervals in the Z-direction. We will call such bases Soria bases, because the $(t, 1/t, s)$-basis is a model case. An understanding of the behavior of these relatively simple bases would be a significant step forward towards the understanding of the general situation.

There are examples of Soria basis of weak type (1,1), of weak type $L\log^+ L$, and of weak type $L\log^2 L$. The purpose of the current note is to introduce a simple geometric property which implies $L\log^2 L$ weak type for Soria bases (compare with $L\log^+ L$ for the $(t, 1/t, s)$-basis). Note, that the projection basis for $(t, 1/t, s)$ is the $(t, 1/t)$-basis, which consists only of incomparable rectangles; *an intersection of any two such rectangles does not belong to the basis.*

A behavior something opposite to this is specified by the following property:

$$\forall k > 1\ \exists R_1, \ldots, R_k \in {B_0}^*\ \forall i \neq j, \qquad (R_i \not\sim R_j)\ \&\ (R_i \cap R_j \in {B_0}^*). \qquad \text{(is)}$$

Here ${B_0}^*$ denotes rectangles from B^* with the left lower vertices at the origin.

Theorem. *Let $\mathcal{B}$ be a Soria basis with* is-*property. Then for any $0 < \lambda < 1$ there exists a set E such that*

$$|\{M_{\mathcal{B}}(\chi_E) > \lambda\}| \geq C \int \frac{\chi_E}{\lambda} \log^2 \frac{\chi_E}{\lambda}\, dx$$

with some constant C independent of E and λ.

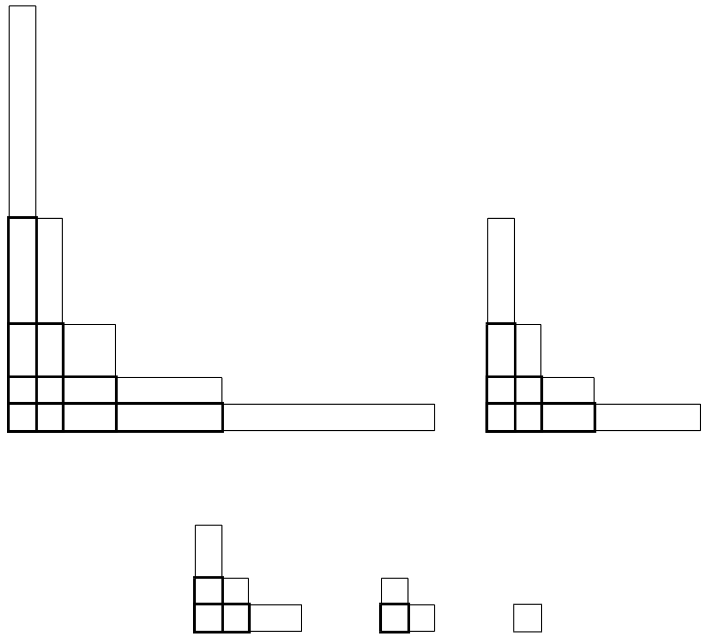
Proof. For a two dimensional rectangle R let $\mathrm{pr}_x(R)$ and $\mathrm{pr}_y(R)$ denote the projections of R onto the X and Y axis, respectively. The (is) property implies that

$$\forall k \geq 1\ \exists R_q^j \in B^*\ (1 \leq j \leq k,\ 1 \leq q \leq j),$$
$$R_q^k \not\sim R_p^k,\ 1 \leq q \neq p \leq k,$$
$$\mathrm{pr}_x(R_q^{j-1}) = \mathrm{pr}_x(R_q^j), \quad \mathrm{pr}_y(R_q^{j-1}) = \mathrm{pr}_y(R_{q+1}^j), \quad q = 1, \ldots, j-1.$$

Geometrically, this means that the stair-shaped set

$$X_j = \bigcup_{1 \leq q \leq j} R_q^j$$

contains an imbedding "Russian-dolls" system of staircases, each of which in turn is a union of rectangles from B^* (see picture below).



The characteristic features of these pictures is that each succeeding picture is an entire fragment of the previous one.

Now, for each rectangle R_i^j let H_i^j denotes $\mathrm{pr}_x(R_i^j)$ and V_i^j denotes $\mathrm{pr}_y(R_i^j)$, i.e. $R_i^j = H_i^j \times V_i^j$, Define the set Θ and the family of sets Y^k. Set

$$\Theta^1 = \Big\{ x_1 \in H_k^k : \prod_{q=1}^{k-1} \sum_{s=0}^{2^{m_q - m_{k-1}}-1} \chi_{H_q^k}(x_1 - 2s|H_q^k|) = 1 \Big\},$$

$$\Theta^2 = \Big\{ x_2 \in V_1^k : \prod_{q=2}^{k} \sum_{s=0}^{2^{n_q - n_{1-1}}-1} \chi_{V_q^k}(x_2 - 2s|V_q^k|) = 1 \Big\}$$

and $\Theta \equiv \Theta^1 \times \Theta^2$.

From geometric reasoning, it is clear that $|\Theta^1| = 2^{1-k}|H_k^k|$, $|\Theta^2| = 2^{1-k}|V_1^k|$, and so $|\Theta| = 2^{2-2k}|H_k^k| \cdot |V_1^k|$.

Next, for fixed $1 \le i \le j \le k$ let $Y_k^{1,k} = H_k^k$, $Y_1^{2,k} = V_1^k$,

$$Y_i^{1,j} = \Big\{ x_1 \in H_k^k : \prod_{q=i}^{k-1} \sum_{s=0}^{2^{m_q - m_{k-1}}-1} \chi_{H_q^k}(x_1 - 2s|H_q^k|) = 1 \Big\},$$

$$Y_i^{2,j} = \Big\{ x_2 \in V_1^k : \prod_{q=2}^{k-j+i} \sum_{s=0}^{2^{n_q - n_{1-1}}-1} \chi_{V_q^k}(x_2 - 2s|V_q^k|) = 1 \Big\}$$

and $Y_i^j \equiv Y_i^{1,j} \times Y_i^{2,j}$. From geometric reasoning again,

$$|Y_i^{1,j}| = 2^{-(k-i)}|H_k^k|, \qquad |Y_i^{2,j}| = 2^{-(k-j+i-1)}|V_1^k|,$$

and hence

$$|Y_i^j| = 2^{-2k+j+1}|H_k^k| \cdot |V_1^k| \qquad (1 \le i \le j \le k).$$

Further, from the definition of the sets $Y_i^{1,j}$ and $Y_i^{2,j}$ it follows that $Y_i^{1,j}$ is a union of translates of the intervals H_i^k, while $Y_i^{2,j}$ is a union of translates of the intervals V_{k-j+i}^k. By the assumptions of the theorem

$$H_i^k = H_i^{k-1} = \cdots = H_i^{j+1} = H_i^j$$

and

$$V_q^k = V_{q-1}^{k-1} = \cdots = V_{q-(k-j)}^j,$$

which upon substituting $k-j+i$ for q yields

$$V_{k-j+i}^k = V_i^j.$$

Thus, $Y_i^{1,j}$ consists of translated intervals H_i^j, and $Y_i^{2,j}$ consists of translated intervals V_i^j. Consequently, Y_i^j consists of translated rectangles R_i^j, i.e. for every $(x_1, x_2) \in Y_i^j$ there is a translation τ such that $\tau(R_i^j) \ni (x_1, x_2)$. Similarity considerations show that

$$\frac{|\tau(R_i^j) \cap \Theta|}{|\tau(R_i^j)|} = \frac{|R_i^j \cap \Theta|}{|R_i^j|} = \frac{|Y_i^j \cap \Theta|}{|Y_i^j|}.$$

Further, the definitions of Θ^1 and Θ^2 imply that

$$\Theta^1 \subset Y_i^{1,j}, \qquad \Theta^2 \subset Y_i^{2,j},$$

hence $\Theta \subset Y_i^j$ and

$$\frac{|Y_i^j \cap \Theta|}{|Y_i^j|} = \frac{|\Theta|}{|Y_i^j|} = 2^{1-j} \qquad (1 \le i \le j \le k).$$

Now set

$$U \equiv \Theta \times [0,1], \qquad Z_i^j \equiv Y_i^j \times [0, 2^{k-j}], \qquad I_i^j \equiv R_i^j \times [0, 2^{k-j}] \qquad (1 \le i \le j \le k).$$

Obviously, the I_i^j are pair-wise incomparable three dimensional rectangles with dyadic side lengths, so if we set

$$W = \bigcup_{j=1}^{k} \bigcup_{i=1}^{j} Z_i^j$$

then

$$|W| \sim \sum_{j=1}^{k} \sum_{i=1}^{j} |Z_i^j| = \sum_{j=1}^{k} \sum_{i=1}^{j} 2^{k-j}|Y_i^j| = \sum_{j=1}^{k} \sum_{i=1}^{j} 2^{k-j}2^{j-1}|\Theta| = 2^{k-2}k(k+1)|U|.$$

By the above considerations, for every $(x_1, x_2, x_3) \in W$ there is rectangle I_i^j and a translation $\bar\tau$ such that $(x_1, x_2, x_3) \in \bar\tau(I_i^j)$ and

$$\bar\tau(I_i^j) = \tau(R_i^j) \times [0, 2^{k-j}],$$

where τ was defined above. Hence,

$$\frac{|\bar\tau(I_i^j) \cap U|}{|\bar\tau(I_i^j)|} = \frac{|\tau(R_i^j) \cap \Theta|}{2^{k-j}|\tau(R_i^j)|} = \frac{|R_i^j \cap \Theta|}{2^{k-j}|R_i^j|} = \frac{2^{1-j}}{2^{k-j}} = 2^{1-k}.$$

These estimates show that

$$W \subset \{x : M_{B^*}(\chi_U)(x) \ge 2^{1-k}\},$$

$$|\{x : M_{B^*}(\chi_U)(x) \ge 2^{1-k}\}| \ge |W| \ge k^2 2^{k-2}|U| \ge 1/2 \int \frac{\chi_U}{2^{1-k}} \log^2 \frac{\chi_U}{2^{1-k}}\, dx,$$

which is reverse to the $L \log^2 L$ estimate.

This completes the proof of the Theorem.

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Department of Mathematics, DePaul University, Chicago, Illinois 60614
E-mail address: astokolo@math.depaul.edu