MONGE–AMPÈRE EQUATIONS AND BELLMAN FUNCTIONS: THE DYADIC MAXIMAL OPERATOR

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Abstract. We find explicitly the Bellman function for the dyadic maximal operator on L^p as the solution of a Bellman PDE of Monge–Ampère type. This function has been previously found by A. Melas [M] in a different way, but it is our PDE-based approach that is of principal interest here. Clear and replicable, it holds promise as a unifying template for past and current Bellman function investigations.

1. Introduction

For a locally integrable function g on $\mathbb{R}^n$ and a set $E \subset \mathbb{R}^n$ with $|E| \neq 0$, let $\langle g \rangle_E = \frac{1}{|E|} \int_E g$ be the average of g over E. Let $p > 1$ and $q > 1$ be conjugate exponents, i.e. $p^{-1} + q^{-1} = 1$. Let φ be a nonnegative locally L^p-function on $\mathbb{R}^n$. Fix a dyadic lattice D on $\mathbb{R}^n$ and consider the dyadic maximal operator

$$M\varphi(x) = \sup_{I \ni x; I \in D} \langle \varphi \rangle_I .$$

Following F. Nazarov and S. Treil [NT], we define the Bellman function for $M\varphi$

$$\boldsymbol{B}(f, F, L) = \sup_{0 \leq \varphi \in L^p_{\mathrm{loc}}(\mathbb{R}^n)} \left\{ \langle (M\varphi)^p \rangle_Q : \langle \varphi \rangle_Q = f; \langle \varphi^p \rangle_Q = F; \sup_{R \supset Q} \langle \varphi \rangle_R = L \right\}. \tag{1.1}$$

Observe that $\boldsymbol{B}$ is independent of Q and well-defined on the domain

$$\Omega = \{(f, F, L) : 0 < f \leq L; f^p \leq F\}.$$

Finding $\boldsymbol{B}$ will, among other things, provide a sharp refinement of the Hardy–Littlewood–Doob maximal inequality

$$\|M\varphi\|_p \leq q \|\varphi\|_p. \tag{1.2}$$

In [NT], the authors show that $\boldsymbol{B}(f, F, L) \leq q^p F - pqfL^{p-1} + pL^p$, which implies (1.2). A. Melas in [M], using deep combinatorial properties of the operator M and without relying on the Bellman PDE, finds $\boldsymbol{B}$ explicitly. In contrast, we develop a boundary value problem of Monge–Ampère type that $\boldsymbol{B}$ must satisfy (assuming sufficient differentiability) and solve it, producing the function from [M]. Our approach has been used as the foundation of several recent Bellman function results. We first restrict our attention to the one-dimensional case and then show that the Bellman function does not depend on dimension.

2000 *Mathematics Subject Classification.* Primary 42A05, 42B35.

Key words and phrases. explicit Bellman function, Bellman PDE, Monge–Ampère equation, dyadic maximal operator.

L. Slavin's research supported in part by the NSF (grant DMS-0701254).

A. Stokolos' research supported in part by a grant from the Faculty Research and Development Fund of the College of Liberal Arts and Sciences of DePaul University.

V. Vasyunin's research supported in part by RFBR (grant 05-01-00925).

2. Finite-differential and differential properties of $\boldsymbol{B}$

Let Q be an interval and Q_-, Q_+ its left and right halves, respectively. Let $(f_\pm, F_\pm) = (f_{Q_\pm}, F_{Q_\pm})$, $(f, F) = ((f_-, F_-) + (f_+, F_+))/2$. Taking suprema in the identity

$$\langle (M\varphi)^p \rangle_Q = \frac{1}{2}\langle (M\varphi)^p \rangle_{Q_-} + \frac{1}{2}\langle (M\varphi)^p \rangle_{Q_+}$$

over all φ with appropriate averages, we obtain

$$\boldsymbol{B}(f, F, L) \geq \frac{1}{2}\boldsymbol{B}(f_-, F_-, \max\{f_-, L\}) + \frac{1}{2}\boldsymbol{B}(f_+, F_+, \max\{f_+, L\}). \tag{2.1}$$

Any function B satisfying this pseudo-concavity property on Ω will be a majorant of the true Bellman function. The following theorem phrases this condition in a differential form.

Theorem 2.1. *Let $z = (f, F)$. Assuming sufficient smoothness on the Bellman function B, condition* (2.1) *holds for all admissible triples if and only if*

$$\det\left(\frac{\partial^2 B}{\partial z^2}\right) = 0,\ B_{ff} \leq 0,\ B_L \geq 0 \text{ on } \Omega; \quad 2B_{fL} + B_{LL} \leq 0,\ B_L = 0 \text{ when } f = L. \tag{2.2}$$

3. Homogeneity, boundary value problem, solution

We reduce the order of the PDE in (2.2) by using the multiplicative homogeneity of $\boldsymbol{B}$: $\boldsymbol{B}(f, F, L) = L^p\boldsymbol{B}(f/L, F/L^p, 1) \overset{\text{def}}{=} L^pG(x, y)$, where $x = f/L, y = F/L^p$. In addition, $F = f^p$ only for functions that are constant on Q, so $\boldsymbol{B}(f, f^p, L) = L^p$, meaning $G(x, x^p) = 1$. Coupling this with the first and the last conditions in (2.2), we get a boundary value problem for G on the domain $\{(x, y)\,|\, 0 < x \leq 1; x^p \leq y\}$:

$$G_{xx}G_{yy} = G_{xy}^2; \qquad G(x, x^p) = 1; \quad pG(1, y) = G_x(1, y) + pyG_y(1, y). \tag{3.1}$$

We look for the solution of the Monge–Ampère equation (3.1) in the general parametric form

$$G(x, y) = tx + f(t)y + g(t); \qquad x + f'(t)y + g'(t) = 0. \tag{3.2}$$

Fix a value of t, i.e. fix one of the straight-line trajectories in (3.2). Let $(u(t), u^p(t))$ be the point where that trajectory intersects the lower boundary $y = x^p$. We have

$$G(u, u^p) = tu(t) + f(t)u^p(t) + g(t) = 1; \qquad u(t) + f'(t)u^p(t) + g'(t) = 0.$$

Differentiating the first equation and using the second one, we get, after some algebra, $f = -t/(pu^{p-1})$, $g = 1 - tu/q$. Assume now that the trajectory intersects the right boundary $x = 1$ at the point $(1, v(t))$. Then $G(1, v) = t + fv + g$. On the other hand, parametrization (3.2) implies $G_x = t$, $G_y = f(t)$ and so the second boundary condition in (3.1) becomes $G(1, v) = \frac{t}{p} + fv$. This gives $g = -t/q$, allowing us to express $t = q/(u-1)$. Simplifying, we obtain a complete solution of the form (3.2):

$$G(x, y) = \frac{y}{u^p}; \qquad x - \frac{qu - 1}{qu^p}y - \frac{1}{q} = 0. \tag{3.3}$$

In terms of the original variables, we get a Bellman function candidate near the boundary $f = L$:

$$B(f, F, L) = Fu^{-p}\left(f/L, F/L^p\right). \tag{3.4}$$

4. From the candidate to the true function

4.1. $B \geq \boldsymbol{B}$. One can readily verify that the rest of conditions (2.2) are satisfied by the candidate (3.4). Therefore, property (2.1) holds and one can perform the Bellman induction: take any non-negative function $\varphi \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ and an interval $Q_0 \in D$. For an interval $Q \subset Q_0, Q \in D$, let $X_Q = (f_Q, F_Q, L_Q)$ with f, F, and L defined as in (1.1). Then

$$\begin{aligned} B(f_{Q_0}, F_{Q_0}, L_{Q_0}) &\geq \frac{1}{2}B(X_{(Q_0)_-}) + \frac{1}{2}B(X_{(Q_0)_+}) \\ &\geq \frac{1}{|Q_0|}\sum_{Q\subset Q_0,\ |Q|=2^{-n}|Q_0|}|Q|B(X_Q) \geq \frac{1}{|Q_0|}\sum_{Q\subset Q_0,\ |Q|=2^{-n}|Q_0|}|Q|L_Q^p \qquad (4.1)\\ &= \frac{1}{|Q_0|}\sum_{Q\subset Q_0,\ |Q|=2^{-n}|Q_0|}|Q|(\sup_{R\supset Q}\langle\varphi\rangle_R)^p \longrightarrow \langle(M\varphi)^p\rangle_{Q_0},\ \text{as } n\to\infty. \end{aligned}$$

Here we have used that $B \geq L^p$. Taking supremum on the right over all φ with the above X_{Q_0} we get $B \geq \boldsymbol{B}$.

4.2. $B \leq \boldsymbol{B}$. To get the reverse inequality, we need to construct, for every point $(f, F, L) \in \Omega$, a sequence of nonnegative functions on $(0,1)$, $\{\varphi_n\}$, so that

$$\lim_{n\to\infty}\langle(M\varphi_n)^p\rangle_{(0,1)} \geq B(f, F, L).$$

To do this, we use the trajectories $t = const$ of the Monge–Ampère equation from Section 3. In the original variables, this gives

$$f = \frac{L}{q} + AF. \qquad (4.2)$$

On the boundary $f = L$ going along these trajectories yields the extremal sequence

$$\varphi_n(t) = \begin{cases} \alpha_n L & 0 < t < 2^{-n} \\ \varphi_n(2^k t - 1) & 2^{-k} < t < 2^{-k+1}, k = 2, ..., n \\ \beta_n \varphi_n(2t-1) & \frac{1}{2} < t < 1. \end{cases} \qquad (4.3)$$

The definition is understood recursively, whereby the function is defined on a portion of $(0,1)$, then on the same portion of the remaining part, and so on. The numbers α_n and β_n are chosen so that $\langle\varphi_n\rangle_{(0,1)} = L$ and $\langle\varphi_n^p\rangle_{(0,1)} = F$. This means

$$\frac{1}{2^n}\alpha_n + \frac{1}{2}\beta_n = \frac{1}{2^n} + \frac{1}{2};\ \ \frac{1}{2^n}\alpha_n^p + \frac{1}{2}\beta_n^p\frac{F}{L^p} = \left(\frac{1}{2^n} + \frac{1}{2}\right)\frac{F}{L^p}.$$

One can show that $\alpha_n M\varphi_n \geq \varphi_n$ and $\alpha_n \to u(1, F/L^p)$ with u defined by (3.3). Therefore,

$$\lim_{n\to\infty}\langle(M\varphi_n)^p\rangle_{(0,1)} \geq \lim_{n\to\infty}\frac{1}{\alpha_n^p}\langle\varphi_n^p\rangle_{(0,1)} = \lim_{n\to\infty}\frac{F}{\alpha_n^p} = Fu^{-p}(1, F/L^p) = \boldsymbol{B}(L, F, L),$$

which gives $\boldsymbol{B}(L, F, L) \geq B(L, F, L)$.

On the boundary $F = f^p$ the situation is simple: here the only test functions are constants and so $B(f, f^p, L) = \boldsymbol{B}(f, f^p, L) = L^p$. Having constructed the extremal sequences on the two boundaries, we get the extremal sequence at any point (f, F, L) with $f > L/q$ as their weighted dyadic rearrangement built along the unique extremal trajectory of the form (4.2) passing through the point.

One observes, however, that trajectories (4.2) cannot be used with $A < 0$, since they then would intersect the “forbidden” boundary $f = 0$. (It is forbidden because, for a nonnegative function, $f = 0$ implies $F = 0$.) In fact, in the region $0 < f < L/q$, no trajectory can lean either to the left or to the right (the forbidden boundary to the left, the existing extremal trajectory $f = L/q$ to the right). We conclude two things: the trajectories are vertical in this region and the candidate (3.4) no longer works there. However, this is quickly rectified: If $G(x,y) = a(x)y + b(x)$, then $G(x, x^p) = 1$ implies that $G(x,y) = 1 + a(x)(y - x^p)$. Now $G_{xx}G_{yy} - G_{xy}^2 = -(a'(x))^2 = 0$, and $G(1/q, y) = q^p y$ implies that $a(x) = q^p$. Thus we get the unique two-piece Bellman function candidate

$$B(f,F,L) = \begin{cases} Fu^{-p}(f/L, F/L^p) & L < qf \\ L^p + q^p(F - f^p) & L \ge qf. \end{cases} \tag{4.4}$$

(In the notation of [M], $u^{-p}(x,y) = \omega_p((px - p + 1)/y)^p$.) This B still satisfies (2.1). Therefore, Bellman induction (4.1) works. We now need an extremal sequence proving that $\boldsymbol{B} \ge B$ in the region $L \ge qf$. There is a unique extremal trajectory passing through each point of the region. However, the trajectory is vertical and so intersects the boundary of Ω at a single point; as a result we cannot use a weighted average of boundary extremal sequences like we just did for the region $L > f/q$. We deal with it by tilting the trajectory slightly to the right, which produces a (distant) second boundary point, at the boundary $f = L$. This lets us use the extremal sequence φ_n from (4.3), while simultaneously reducing the tilt. Namely, fix (f,F,L) and $k \ge 1$. Define γ_k and F_k so that $L - \gamma_k = 2^k(f - \gamma_k)$ and $F_k - \gamma_k^p = 2^k(F - \gamma_k^p)$. (Observe that $\gamma_k \to f$ and $F_k \to \infty$.) Using (4.3), form a sequence $\{\varphi_{k,n}\}_{n=1}^\infty$ with $\langle \varphi_{k,n} \rangle_{(0,1)} = L$ and $\langle \varphi_{k,n}^p \rangle_{(0,1)} = F_k$, so that $\langle (M\varphi_{k,n})^p \rangle_{(0,1)} \to B(L, F_k, L)$, as $n \to \infty$. Let

$$\psi_{k,n}(t) = \begin{cases} \varphi_{k,n}(2^k t) & 0 < t < 2^{-k} \\ \gamma_k & 2^{-k} < t < 1 \\ 2L - f & 1 < t < 2. \end{cases}$$

Direct computation shows that $\langle \psi_{k,n} \rangle_{(0,1)} = f$, $\langle \psi_{k,n}^p \rangle_{(0,1)} = F$, and $\langle \psi_{k,n} \rangle_{(0,2)} = L$. Then

$$\begin{aligned} \langle (M\psi_{k,n})^p \rangle_{(0,1)} &\ge L^p(1 - 2^{-k}) + 2^{-k} \langle (M\varphi_{k,n})^p \rangle_{(0,1)} \\ &\xrightarrow[n\to\infty]{} L^p(1 - 2^{-k}) + 2^{-k} B(L, F_k, L) \\ &\xrightarrow[k\to\infty]{} L^p + (F - f^p)u^{-p}(1, \infty) = L^p + q^p(F - f^p), \end{aligned}$$

5. Several dimensions

It turns out that the Bellman function (1.1),(4.4) is dimension-free. Fix a dyadic cube Q and let $Q_1, ..., Q_{2^n}$ be its dyadic offspring. Then

$$B\left(2^{-n} \sum_{k=1}^{2^n} z_k, L\right) \ge 2^{-n} \sum_{k=1}^{n} B(z_k, \max\{f_k, L\}).$$

Therefore, we can run the induction (4.1) to prove that $B \ge \boldsymbol{B}$. The other direction is shown by a trivial modification of the one-dimensional maximizing sequences. A similar argument can be used to show that the same Bellman function works for the maximal operator on trees, the setting of choice in [M].

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