A NOTE ON THE MAXIMAL GUROV-RESHETNYAK CONDITION

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Dedicated to Professor Bogdan Bojarski on the occasion of his 75th birthday

ABSTRACT. In a recent paper [17] we established an equivalence between the Gurov-Reshetnyak and A_∞ conditions for arbitrary absolutely continuous measures. In the present paper we study a weaker condition called the *maximal* Gurov-Reshetnyak condition. Although this condition is not equivalent to A_∞ even for Lebesgue measure, we show that for a large class of measures satisfying Busemann-Feller type condition it will be self-improving as is the usual Gurov-Reshetnyak condition. This answers a question raised independently by T. Iwaniec and V.I. Kolyada.

1. INTRODUCTION

Throughout the paper, Q_0 will be a bounded cube from $\mathbb{R}^n$, and μ will be a non-negative Borel measure on Q_0 absolutely continuous with respect to Lebesgue measure. For $f \in L_\mu(Q_0)$ and for any subcube $Q \subset Q_0$ set

$$f_{Q,\mu} = \frac{1}{\mu(Q)} \int_Q f(x)d\mu, \quad \Omega_\mu(f;Q) = \frac{1}{\mu(Q)} \int_Q |f(x) - f_{Q,\mu}|d\mu.$$

A function $f \in L_\mu(Q_0)$ is said to belong $BMO(\mu)$ if

$$\sup_{Q \subset Q_0} \Omega_\mu(f;Q) < \infty.$$

Also we recall that the classes $A_p(\mu)$ and $RH_r(\mu), 1 < p, r < \infty$, consist of all non-negative $f \in L_\mu(Q_0)$ for which there exists $c > 0$ such that for all $Q \subset Q_0$,

$$\left(f_{Q,\mu}\right)\left((f^{-1/(p-1)})_{Q,\mu}\right)^{p-1} \leq c \quad \text{and} \quad (f^r)_{Q,\mu} \leq c(f_{Q,\mu})^r,$$

respectively. In the unweighted case (i.e., in the case when μ is Lebesgue measure) these objects were first considered in the classical works by F. John and L. Nirenberg [15], B. Muckenhoupt [20], F. Gehring [9],

and R. Coifman and C. Fefferman [5]. It has been quickly realized that the theory developed in the unweighted case remains true for doubling measures (i.e., for those measures μ for which there exists a constant $c > 0$ such that $\mu(2Q) \leq c\mu(Q)$ for all cubes Q). Only recently, it was shown in the papers by J. Mateu *et al.* [18] and by J. Orobitg and C. Pérez [21] that most of important results concerning $BMO(\mu), A_p(\mu)$ and $RH_r(\mu)$ still hold for any absolutely continuous measure μ (and even for a wider class of measures). First of all, we mention that all these objects are closely related. Namely (see [21]),

$$(1.1) \qquad A_\infty(\mu) \equiv \bigcup_{1<p<\infty} A_p(\mu) = \bigcup_{1<r<\infty} RH_r(\mu),$$

and $\log f \in BMO(\mu)$ for any $f \in A_\infty(\mu)$, and, conversely, given an $f \in BMO(\mu)$, there is $\lambda > 0$ such that $e^{\lambda f} \in A_\infty(\mu)$. Also, conditions expressed in the above definitions represent a kind of so-called self-improving properties. Indeed, if $f \in L_\mu(Q_0)$ belongs to $BMO(\mu)$, then f belongs to $L_\mu^p(Q_0)$ for any $1 < p < \infty$ (see [18]). Moreover, $A_p(\mu) \Rightarrow A_{p-\delta}(\mu)$ and $RH_r(\mu) \Rightarrow RH_{r+\delta}(\mu)$ for some small $\delta > 0$ which may differ in each implication (see [21]).

In the mid 70s, L.G. Gurov and Yu.G. Reshetnyak [10, 11] introduced in analogy with the definition of $BMO(\mu)$ (in the unweighted case) the class $GR_\varepsilon(\mu), 0 < \varepsilon < 2$, which consists of all non-negative $f \in L_\mu(Q_0)$ such that for any $Q \subset Q_0$,

$$(1.2) \qquad \Omega_\mu(f;Q) \leq \varepsilon f_{Q,\mu}.$$

This class has found interesting applications in quasi-conformal mappings and PDE's (see, e.g., [3, 14]). Observe that (1.2) trivially holds for $\varepsilon = 2$, and therefore only the case $0 < \varepsilon < 2$ is of interest. It turned out that (1.2) also represents a kind of self-improving property. It was established in [3, 10, 11, 14, 19, 26] for Lebesgue measure and in [7, 8] for doubling measures that if ε is small enough, namely $0 < \varepsilon < c2^{-n}$, then the $GR_\mu(\varepsilon)$ implies $f \in L_\mu^p(Q_0)$ for some $p > 1$. In [16], it was shown that in the case of $n = 1$ and Lebesgue measure this self-improvement holds for the whole range $0 < \varepsilon < 2$. In a recent paper [17], the authors have established a rather surprising analogue of (1.1), namely for any absolutely continuous μ,

$$(1.3) \qquad A_\infty(\mu) = \bigcup_{0<\varepsilon<2} GR_\varepsilon(\mu).$$

First, this result shows a close relation between the classes $GR_\varepsilon(\mu)$ and $A_p(\mu)$. Second, it follows immediately that for any absolutely

continuous μ and $n \geq 1$ and for all $0 < \varepsilon < 2$ the $GR_\varepsilon(\mu)$ condition implies higher integrability properties of f.

T. Iwaniec and V.I. Kolyada independently asked the authors whether a weaker variant of (1.2):

$$(1.4) \qquad f^{\#}_{\mu,Q_0}(x) \leq \varepsilon M_{\mu,Q_0} f(x) \quad \mu\text{-a.e. in } Q_0$$

has an analogous self-improving property for all $\varepsilon < 2$. Here, as usual,

$$f^{\#}_{\mu,Q_0}(x) = \sup_{Q \ni x, Q \subset Q_0} \Omega_\mu(f;Q) \quad \text{and} \quad M_{\mu,Q_0} f(x) = \sup_{Q \ni x, Q \subset Q_0} |f|_{Q,\mu}.$$

The property expressed in (1.4) we call the *maximal* Gurov-Reshetnyak condition, and denote it by $MGR_\varepsilon(\mu)$.

Observe that the passing to maximal operators is a quite natural and well-known approach to many questions mentioned above. For example, Gehring's approach to the reverse Hölder inequality [9] as well as Bojarski's proof of the Gurov-Reshetnyak Lemma for small ε [3] were based on maximal function estimates. Actually, many papers establishing the self-improving property of $GR_\varepsilon(\mu)$ for small ε contain implicitly the same for $MGR_\varepsilon(\mu)$ (see, e.g., [3, 7, 19]). On the other hand, author's proof of (1.3) cannot be directly generalized to the class $MGR_\varepsilon(\mu)$. Therefore, the question of T.Iwaniec and V.I. Kolyada is of interest for $c < \varepsilon < 2$.

In this paper we show that for a large class of measures, including any doubling measures in $\mathbb{R}^n$ and any absolutely continuous measures in $\mathbb{R}^1$, the maximal Gurov-Reshetnyak condition $MGR_\varepsilon(\mu)$ is self-improving for any $0 < \varepsilon < 2$. The relevant class of measures will be given in the following definition.

Definition 1.1. We say that a measure μ satisfies the Busemann-Feller type condition (BF-condition) if

$$\varphi_\mu(\lambda) \equiv \sup_E \frac{\mu\{x : M_{\mu,Q_0}\chi_E(x) > \lambda\}}{\mu(E)} < \infty \quad (0 < \lambda < 1),$$

where the supremum is taken over all measurable sets $E \subset Q_0$ of positive μ-measure.

In the case of Lebesgue measure and the maximal operator associated with the homothety-invariant differential basis, this condition coincides with the well-known Busemann-Feller density condition (see, e.g., [4] or [12, p. 122]).

Our main result is the following.

Theorem 1.2. *Let μ satisfy the BF-condition, and let $0 < \varepsilon < 2$. Assume that a non-negative $f \in L_\mu(Q_0)$ satisfies the maximal Gurov-Reshetnyak condition $MGR_\varepsilon(\mu)$. Then there is $p_0 > 1$ depending on*

μ, ε and n such that for all $1 \leq p < p_0$ one has

$$(1.5) \qquad (f^p)_{Q_0,\mu} \leq c(f_{Q_0,\mu})^p,$$

where c depends on μ, ε, p and n.

Some comments about this result are in order. By (1.1) and (1.3), the usual Gurov-Reshetnyak condition $GR_\varepsilon(\mu)$ implies (1.5) with any subcube $Q \subset Q_0$ instead of Q_0. Theorem 1.2 shows that although we cannot obtain from the $MGR_\varepsilon(\mu)$ condition such a nice conclusion, we still have a higher integrability result. In fact, $MGR_\varepsilon(\mu)$ is really much weaker than $GR_\varepsilon(\mu)$. Indeed, let $n = 1, Q_0 = (0,1)$, μ is Lebesgue measure, and f, for example, is the characteristic function of the interval $(0, 7/8)$. Then it is easy to see that $f^{\#}_{Q_0,\mu}(x) \leq 1/2$, while $M_{Q_0,\mu}f(x) > 7/8$ for all $x \in (0,1)$. Thus, (1.4) holds for this function with $\varepsilon = 4/7$. However, $f \notin A_\infty(\mu)$, since it is zero on a set of positive measure. Hence, in view of (1.3), (1.2) cannot hold for this f with any $\varepsilon < 2$. This example shows also that the class $GR_\varepsilon(\mu)$ in (1.3) cannot be replaced by $MGR_\varepsilon(\mu)$.

We make several remarks about the BF-condition. It is easy to see that this condition holds provided M_μ has a weak type (p, p) property with respect to μ for some $p > 0$. A standard argument shows that if μ is doubling in $\mathbb{R}^n$, then M_μ is of weak type $(1, 1)$. It is well-known that in the case $n = 1$ the doubling condition can be completely removed; namely in this case, M_μ is of weak type $(1, 1)$ for arbitrary Borel measure μ (see [22]). It has been recently shown in [23] that in the case $n \geq 2$ for a large class of radial (and non-doubling, in general) measures, including, for example, a Gaussian measure, M_μ will be of strong type (p, p) for any $p > 1$. On the other hand, it was mentioned in [25] that there exists μ for which M_μ will not be of strong type (p, p) for any $p > 1$. In Section 3 below we give an example of μ (in the case $n = 2$) for which the BF-condition does not hold.

We would like to emphasize that we still do not know whether the BF-condition in Theorem 1.2 is really necessary. In other words we do not know whether there exist an absolutely continuous measure μ on a cube $Q_0 \subset \mathbb{R}^n, n \geq 2$, and a function $f \in L_\mu(Q_0)$ such that μ does not satisfy the BF-condition, f satisfies (1.4) for some $\varepsilon < 2$ and $f \notin L^p_\mu(Q_0)$ for any $p > 1$.

Let us mention also that Theorem 1.2 gives yet another proof of the Gurov-Reshetnyak Lemma for the whole range of ε if μ is a BF-measure.

The paper is organized as follows. In the next section we prove our main result. Section 3 contains a detailed analysis of the BF-condition.

2. Proof of main result

2.1. **Some auxiliary propositions.** We first recall that the non-increasing rearrangement of a measurable function f on Q_0 with respect to μ is defined by

$$f_\mu^*(t) = \inf\{\alpha > 0 : \mu\{x \in Q_0 : |f(x)| > \alpha\} \le t\} \quad (0 < t < \mu(Q_0)).$$

Set also $f_\mu^{**}(t) = t^{-1}\int_0^t f_\mu^*(\tau)d\tau$. Note that [2, pp. 43, 53]

(2.1)
$$\int_0^t f_\mu^*(\tau)d\tau = \sup_{E \subset Q_0 : \mu(E)=t} \int_E |f|d\mu$$

and

(2.2)
$$\int_{Q_0} |f|^p d\mu = \int_0^{\mu(Q_0)} f_\mu^*(\tau)^p d\tau \quad (p > 0).$$

We will need a local variant of the well-known Herz-type estimate

$$f_\mu^{**}(t) \le c(M_\mu f)_\mu^*(t).$$

In the case of $Q_0 = \mathbb{R}^n$ and Lebesgue measure this result can be found in [2, p. 122]. It was extended to arbitrary absolutely continuous measures μ in [1]. The case of the bounded cube Q_0 requires a slightly modified argument based on the following covering lemma from [18].

Lemma 2.1. *Let E be a subset of Q_0 with $\mu(E) \le \rho\mu(Q_0)$, $0 < \rho < 1$. Then there exists a sequence $\{Q_i\}$ of cubes contained in Q_0 such that*

 (i) $\mu(Q_i \cap E) = \rho\mu(Q_i)$;

 (ii) $\bigcup_i Q_i = \bigcup_{k=1}^{B_n} \bigcup_{i \in F_k} Q_i$, *where each of the family $\{Q_i\}_{i \in F_k}$ is formed by pairwise disjoint cubes and a constant B_n depends only on n;*

 (iii) $E' \subset \cup_i Q_i$, *where E' is the set of μ-density points of E.*

Proposition 2.2. *For any $f \in L_\mu(Q_0)$ we have*

(2.3)
$$f_\mu^{**}(t) \le c_n(M_{\mu,Q_0}f)_\mu^*(t) \quad (0 < t < \mu(Q_0)),$$

where c_n depends only on n.

Proof. Let $\Omega = \{x \in Q_0 : M_{\mu,Q_0}f(x) > (M_{\mu,Q_0}f)_\mu^*(t)\}$. Then for some δ_0 and for any $\delta < \delta_0$ we have $\mu(\Omega) \le t \le (1-\delta)\mu(Q_0)$. Fix such a δ and apply Lemma 2.1 to the set Ω and number $\rho = 1 - \delta$. We get a sequence $\{Q_i\}$ satisfying properties (i)-(iii) of the lemma. It follows easily from (i) that $\mu(Q_i \cap \Omega^c) > 0$, and hence, $|f|_{Q_i,\mu} \le (M_{\mu,Q_0}f)_\mu^*(t)$. From this and from properties (i)-(iii) we obtain

$$\int_\Omega |f|d\mu \le \int_{\cup_i Q_i} |f|d\mu \le \sum_{k=1}^{B_n} \sum_{i \in F_k} \mu(Q_i)|f|_{Q_i,\mu} \le \frac{B_n}{1-\delta}t(M_{\mu,Q_0}f)_\mu^*(t).$$

Letting $\delta \to 0$ yields $\int_\Omega |f| d\mu \le B_n t (M_{\mu,Q_0} f)^*_\mu(t)$. Therefore, for any measurable set $E \subset Q_0$ with $\mu(E) = t$,

$$\int_E |f| d\mu \le \int_{E \setminus \Omega} |f| d\mu + \int_\Omega |f| d\mu \quad \le \quad (\mu(E \setminus \Omega) + B_n t)(M_{\mu,Q_0} f)^*_\mu(t)$$

$$\le \quad (B_n + 1) t (M_{\mu,Q_0} f)^*_\mu(t).$$

Taking the supremum over all $E \subset Q_0$ with $\mu(E) = t$ and using (2.1) completes the proof. $\qquad\square$

Given a measurable f, define the local maximal function $m_{\lambda,\mu} f$ (cf. [24]) by

$$m_{\lambda,\mu} f(x) = \sup_{Q \ni x, Q \subset Q_0} \left(f \chi_Q\right)^*_\mu \left(\lambda \mu(Q)\right) \quad (0 < \lambda < 1).$$

Proposition 2.3. *Suppose that μ satisfies the BF-condition. Then for any measurable f,*

$$(2.4) \qquad (m_{\lambda,\mu} f)^*_\mu(t) \le f^*_\mu\left(t/\varphi_\mu(\lambda)\right) \quad (0 < t < \mu(Q_0)).$$

Proof. It follows from the definitions that

$$\{x \in Q_0 : m_{\lambda,\mu} f(x) > \alpha\} = \{x \in Q_0 : M_{\mu,Q_0} \chi_{\{|f| > \alpha\}}(x) > \lambda\}.$$

Therefore,

$$\mu\{x \in Q_0 : m_{\lambda,\mu} f(x) > \alpha\} \le \varphi_\mu(\lambda) \mu\{x \in Q_0 : |f(x)| > \alpha\},$$

which is equivalent to (2.4). $\qquad\square$

Lemma 2.4. ([20, Lemma 4].) *Let h be a non-negative and non-increasing function on the interval $[0, a]$, and assume that*

$$\frac{1}{s} \int_0^s h(\tau) d\tau \le D h(s) \quad (0 < s < a/r)$$

for some $D, r > 1$. Then if $1 \le p < D/(D-1)$,

$$\int_0^a h^p(\tau) d\tau \le c \left(\int_0^a h(\tau) d\tau \right)^p,$$

where c depends on r, p and D.

Actually, this lemma was proved in [20] with $r = 20$ but exactly the same argument works for any $r > 1$.

2.2. **Proof of Theorem 1.2.** Choose some constants $\alpha, \lambda \in (0,1)$ such that $(1-\alpha)\lambda > \varepsilon/2$. Take an arbitrary $\delta \in (0, 1-\lambda)$. Given a cube $Q \subset Q_0$, set $E_Q = \{x \in Q : f(x) > f_{Q,\mu}\}$, and let

$$\mathcal{Q} = \{Q \subset Q_0 : \mu(E_Q) \geq \alpha\mu(Q)\}.$$

Observe that if $Q \subset \mathcal{Q}$, then $f_{Q,\mu} \leq (f\chi_Q)^*_\mu(\alpha\mu(Q))$. Therefore,

$$(2.5) \qquad M_{\mu,Q_0}f(x) \leq \max\left(m_{\alpha,\mu}f(x), \sup_{Q \ni x, Q \not\subset \mathcal{Q}} f_{Q,\mu}\right).$$

Assume that $Q \not\subset \mathcal{Q}$. Set $E_Q^c = Q \setminus E_Q$,

$$A_1(Q) = \left\{x \in E_Q^c : f_{Q,\mu} - f(x) > \left((f_{Q,\mu} - f)\chi_{E_Q^c}\right)^*_\mu\left(\lambda\mu(E_Q^c)\right)\right\}$$

and

$$A_2(Q) = \left\{x \in E_Q^c : f(x) > \left(f\chi_{E_Q^c}\right)^*_\mu\left((1-\lambda-\delta)\mu(E_Q^c)\right)\right\}.$$

Then $\mu\big(A_1(Q) \cup A_2(Q)\big) \leq (1-\delta)\mu(E_Q^c)$ and $\mu(E_Q^c) \geq (1-\alpha)\mu(Q)$. Therefore we obtain

$$f_{Q,\mu} \leq \inf_{y \in E_Q^c \setminus \{A_1(Q) \cup A_2(Q)\}} \left((f_{Q,\mu} - f(y)) + f(y)\right)$$

$$\leq \left((f_{Q,\mu} - f)\chi_{E_Q^c}\right)^*_\mu\left(\lambda\mu(E_Q^c)\right) + \left(f\chi_{E_Q^c}\right)^*_\mu\left((1-\lambda-\delta)\mu(E_Q^c)\right)$$

$$\leq \frac{\mu(Q)}{\mu(E_Q^c)}\frac{1}{2\lambda}\frac{2}{\mu(Q)}\int_{E_Q^c}(f_{Q,\mu} - f)d\mu + \left(f\chi_{E_Q^c}\right)^*_\mu\left((1-\lambda-\delta)\mu(E_Q^c)\right)$$

$$\leq \frac{1}{2\lambda(1-\alpha)}\Omega_\mu(f;Q) + (f\chi_Q)^*_\mu((1-\lambda-\delta)(1-\alpha)\mu(Q)).$$

This along with the maximal Gurov-Reshetnyak condition (1.4) yields

$$\sup_{Q \ni x, Q \not\subset \mathcal{Q}} f_{Q,\mu} \leq \frac{\varepsilon}{2\lambda(1-\alpha)}M_{\mu,Q_0}f(x) + m_{\gamma,\mu}f(x),$$

where $\gamma = (1-\lambda-\delta)(1-\alpha)$. From this and from (2.5) we obtain

$$M_{\mu,Q_0}f(x) \leq cm_{\alpha',\mu}f(x),$$

where $c = \frac{2\lambda(1-\alpha)}{2\lambda(1-\alpha)-\varepsilon}$ and $\alpha' = \min(\alpha, \gamma)$. Taking the rearrangements of both parts and using Propositions 2.2 and 2.3, we have

$$f_\mu^{**}(t) \leq cf_\mu^*(t/\varphi_\mu(\alpha')) \quad (0 < t < \mu(Q_0)).$$

This implies easily

$$f_\mu^{**}(t) \leq c\varphi_\mu(\alpha')f_\mu^*(t) \quad (0 < t < \mu(Q_0)/\varphi_\mu(\alpha')),$$

which along with (2.2) and Lemma 2.4 completes the proof.

3. On the BF-condition

First of all, we observe that we do not know whether for some absolutely continuous μ the function φ_μ can take both finite and infinite values. The following proposition represents only a partial answer to this question.

Proposition 3.1. *There exists a constant $\gamma < 1$ depending only on n such that if $\varphi_\mu(\gamma) < \infty$, then $\varphi_\mu(\lambda) < \infty$ for all $0 < \lambda < 1$.*

Proof. Given a set $E \subset Q_0$ and $0 < \lambda < 1$, let $E_\lambda = \{x \in Q_0 : M_{\mu,Q_0}\chi_E(x) > \lambda\}$. By Proposition 2.2, for any $E \subset Q \subset Q_0$,

$$\min(1, \mu(E)/t) \leq c_n (M_{\mu,Q}\chi_E)^*_\mu(t)$$

or, equivalently,

$$\frac{\mu(E)}{c_n\lambda} \leq \mu\{x \in Q : M_{\mu,Q}\chi_E(x) > \lambda\} \leq \mu(Q \cap E_\lambda).$$

Hence,

$$M_{\mu,Q_0}\chi_E(x) \leq c_n\lambda M_{\mu,Q_0}\chi_{E_\lambda}(x) \quad (x \in Q_0),$$

which yields

$$(3.1) \quad \mu\{x \in Q_0 : M_{\mu,Q_0}\chi_E > c_n\xi\lambda\} \leq \mu\{x \in Q_0 : M_{\mu,Q_0}\chi_{E_\lambda} > \xi\}.$$

Therefore,

$$\varphi_\mu(c_n\lambda\xi) \leq \varphi_\mu(\lambda)\varphi_\mu(\xi) \quad (\lambda, \xi \in (0,1) : \lambda\xi < 1/c_n).$$

This clearly implies the desired result if $0 < \lambda < 1/c_n$. The case $1/c_n < \lambda < 1$ follows from the monotonicity of $\varphi_\mu(\lambda)$. $\square$

Remark 3.2. The last proposition means that Theorem 1.2 still holds if one relaxes the BF-condition to

$$\mu\{x \in Q_0 : M_{\mu,Q_0}\chi_E > \gamma\} \leq c\mu(E) \quad \forall E \subset Q_0$$

with some $0 < \gamma < 1$. Note that a similar condition with $\gamma = 1/2$ for the directional maximal operator appeared in [6] (see also [12, p. 372]), where it was called a Tauberian condition.

Remark 3.3. Inequality (3.1) is a full analogue of the Lemma from [13], which was proved there in a different context.

We give now an example of the absolutely continuous measure μ on the unit cube $(0,1)^2$ which does not satisfy the BF-condition.

Let $\delta, L > 0$ and

$$d\mu = \big(\delta\chi_{(-L,0)^2}(x,y) + \chi_{(0,L)^2}(x,y)\big)dxdy.$$

Given a point $(x, y) \in (0, L)^2$, let Q be a minimal cube contained in $(-2L, 2L)^2$ and containing (x, y) and the cube $(-L, 0)^2$. Then $\mu(Q) = xy + \delta L^2$. Hence, setting $E = (-L, 0)^2$, we get

$$(3.2) \qquad M_{\mu,(-2L,2L)^2} \chi_E(x, y) \geq \frac{\mu(Q \cap E)}{\mu(Q)} = \frac{\delta L^2}{xy + \delta L^2}.$$

Denote $c_\lambda = \frac{1}{\lambda} - 1$. Assuming $\delta c_\lambda < 1$, we get, by (3.2),

$$\frac{\mu\{M_{\mu,(-2L,2L)^2} \chi_E(x, y) > \lambda\}}{\mu(E)} \geq \frac{\mu\{(x, y) \in (0, L)^2 : xy < \delta L^2 c_\lambda\}}{\delta L^2}$$

$$(3.3) \geq c_\lambda \int_{\delta c_\lambda L}^{L} \frac{dx}{x} = c_\lambda \log \frac{1}{\delta c_\lambda}.$$

Choose now a sequence of cubes $\{Q_i\}$ such that the cubes $2Q_i$ are pairwise disjoint and $\cup_{i=1}^{\infty} 2Q_i \subset (0, 1)^2$. We divide each cube Q_i into four equal quadrants, and let Q_i' and Q_i'' be the first and the third quadrants respectively. Let $\{\delta_i\}$ be a sequence of positive numbers such that $\delta_i \to 0$ as $i \to \infty$. Set

$$d\mu = \sum_{i=1}^{\infty} \big(\delta_i \chi_{Q_i''}(x, y) + \chi_{Q_i'}(x, y)\big) dx dy.$$

Given a $\lambda \in (0, 1)$, there is an N such that $\delta_i c_\lambda < 1$ for all $i \geq N$. Hence, by (3.3),

$$(3.4) \qquad \frac{\mu\{M_{\mu,(0,1)^2} \chi_{Q_i''} > \lambda\}}{\mu(Q_i'')} \geq c_\lambda \log \frac{1}{c_\lambda \delta_i} \quad (i \geq N).$$

This shows that μ does not satisfy the BF-condition, since the right-hand side of (3.4) tends to ∞ as $i \to \infty$.

In conclusion we give one more proposition concerning the function φ_μ which is probably of some independent interest. We recall that the operator M_μ is said to be of restricted weak type (p, p) if there exists $c > 0$ such that

$$(3.5) \qquad \varphi_\mu(\lambda) \leq c\lambda^{-p} \quad (0 < \lambda < 1).$$

By the well-known interpolation theorem of E.M. Stein and G. Weiss [2, p. 233], (3.5) implies the strong type (q, q) of M_μ for $q > p$. The following proposition shows first that a slightly better estimate than (3.5) allows us to get the strong type (p, p), and, second, it yields a very simple proof of the Stein-Weiss theorem for M_μ.

Proposition 3.4. *If $\int_0^1 \varphi_\mu(\lambda)^{1/q}d\lambda < \infty$, then M_μ is bounded on L_μ^q and*

$$\|M_\mu f\|_{L_\mu^q} \leq \Big(\int_0^1 \varphi_\mu(\lambda)^{1/q}d\lambda \Big) \|f\|_{L_\mu^q} \quad (1 \leq q < \infty).$$

Proof. By (2.2),

$$\frac{1}{\mu(Q)} \int_Q |f|d\mu = \int_0^1 (f\chi_Q)_\mu^*(\lambda\mu(Q))d\lambda,$$

and thus,

$$M_\mu f(x) \leq \int_0^1 m_{\lambda,\mu}f(x)d\lambda.$$

Applying Minkowski's inequality along with Proposition 2.3 yields

$$\|M_\mu f\|_{L_\mu^q} \leq \int_0^1 \|m_{\lambda,\mu}f\|_{L_\mu^q}d\lambda \leq \Big(\int_0^1 \varphi_\mu(\lambda)^{1/q}d\lambda \Big)\|f\|_{L_\mu^q},$$

as required. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

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