

L^p estimates for the maximal functions

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Dedicated to Olga Vladimirovna Voskoboynik
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▶ Hardy–Littlewood–Doob–Cox

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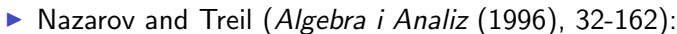
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$$\mathbf{B}(f, F, L) = \sup_{0 \leq \varphi \in L^p_{\text{loc}}(\mathbb{R})} \left\{ \langle (M\varphi)^p \rangle_Q : \langle \varphi \rangle_Q = f; \right. \\ \left. \langle \varphi^p \rangle_Q = F; \sup_{R \supset Q} \langle \varphi \rangle_R = L \right\}.$$

L^p estimates for the maximal functions

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- ▶ In particular case $p = 2$:

$$\mathbf{B}(f, F, L) = \begin{cases} (\sqrt{F} + \sqrt{F + L^2 - 2Lf})^2 & \text{for } L \leq 2f \\ L^2 + 4(F - f^2) & \text{for } L \geq 2f \end{cases}$$

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- ▶ Slavin, Stokolos and Vasyunin (*C. R. Math. Acad. Sci.* (2008), 585-588) suggested the alternative approach based on application of the Monge-Ampère PDE.

Interpolation via Bellman Function



$$\mathbf{B}(x_1, x_2) = \inf_{\varphi \geq 0} \{ \langle \sqrt{\varphi} \rangle_Q : \langle \varphi \rangle_Q = x_1; \langle \varphi^2 \rangle_Q = x_2 \}.$$

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$$\mathbf{B}(cx_1, c^2x_2) = c\mathbf{B}(x_1, x_2) \Rightarrow \mathbf{B}(x_1, x_2) = \sqrt{x_1} \mathbf{B}\left(1, \frac{x_2}{x_1^2}\right).$$

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$$\mathbf{B}(x_1, x_1^2) = \sqrt{x_1}.$$

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- ▶ The solution $C(t) = t^{-1/2}$ gives

$$B(x_1, x_2) = x_1 \sqrt{\frac{x_1}{x_2}}.$$

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$$B(x_1, x_2) = B\left(\sum_{I \subset [0,1]} |I|(\langle \varphi \rangle_I, \langle \varphi^2 \rangle_I)\right) \leq \sum_{I \subset [0,1]} |I| B(\langle \varphi \rangle_I, \langle \varphi^2 \rangle_I) =$$

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$$\sum_{I \subset [0,1]} |I| \frac{\langle \varphi \rangle_I^{3/2}}{\langle \varphi^2 \rangle_I^{1/2}} \approx \sum_{I \subset [0,1]} \langle \varphi \rangle_I^{1/2} |I| \rightarrow \langle \sqrt{\varphi} \rangle_Q.$$

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- ▶ From Hölder

$$\|\varphi\|_1^4 \leq \|\varphi\|_{\frac{1}{2}} \|\varphi\|_{\frac{3}{2}}^3 \leq \|\varphi\|_{\frac{1}{2}} \|\varphi\|_2^3.$$

Interpolation in Orlicz classes

Theorem (Slavin, S., Vasyunin).

Let

$$\mathbf{B}(x_1, x_2) = \inf_{\varphi \geq 0} \{ \langle \log(1+\varphi) \rangle_Q : \langle \varphi \rangle_Q = x_1; \langle (1+\varphi) \log(1+\varphi) \rangle_Q = x_2 \}.$$

Then

$$\mathbf{B}(x_1, x_2) = \frac{x_2}{1+w},$$

where w is the unique solution of the equation

$$\frac{(1+w) \log(1+w)}{w} = \frac{x_2}{x_1}.$$

in the domain $(1, \infty)$.