L^p estimates for the maximal functions

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Dedicated to Olga Vladimirovna Voskoboynik
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- Nazarov and Treil (*Algebra i Analiz* (1996), 32-162):

$$\mathbf{B}(f, F, L) = \sup_{0 \le \varphi \in L^p_{\text{loc}}(\mathbb{R})} \Big\{ \langle (M\varphi)^p \rangle_Q : \langle \varphi \rangle_Q = f;$$
$$\langle \varphi^p \rangle_Q = F;\ sup_{R \supset Q} \langle \varphi \rangle_R = L \Big\}.$$

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- In particular case $p = 2$:

$$\mathbf{B}(f, F, L) = \begin{cases} (\sqrt{F} + \sqrt{F + L^2 - 2Lf})^2 & \text{for} \quad L \le 2f \\ L^2 + 4(F - f^2) & \text{for} \quad L \ge 2f \end{cases}$$

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- Slavin, Stokolos and Vasyunin (*C. R. Math. Acad. Sci.* (2008), 585-588) suggested the alternative approach based on application of the Monge-Ampére PDE.

Interpolation via Bellman Function

- $$\mathbf{B}(x_1, x_2) = \inf_{\varphi \geq 0} \{ \langle \sqrt{\varphi} \rangle_Q : \langle \varphi \rangle_Q = x_1; \langle \varphi^2 \rangle_Q = x_2 \}.$$

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- $$\mathbf{B}(x_1, x_1^2) = \sqrt{x_1}.$$

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- The solution $C(t) = t^{-1/2}$ gives

$$B(x_1, x_2) = x_1\sqrt{\frac{x_1}{x_2}}.$$

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- $$B(x_1, x_2) = B\left(\sum_{I\subset[0,1]} |I|(\langle\varphi\rangle_I, \langle\varphi^2\rangle_I)\right) \le \sum_{I\subset[0,1]} |I| B(\langle\varphi\rangle_I, \langle\varphi^2\rangle_I) =$$

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- $$\sum_{I\subset[0,1]} |I| \frac{\langle\varphi\rangle_I^{3/2}}{\langle\varphi^2\rangle_I^{1/2}} \approx \sum_{I\subset[0,1]} \langle\varphi\rangle_I^{1/2} |I| \rightarrow \langle \sqrt{\varphi}\rangle_Q.$$

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- From Hölder

$$\|\varphi\|_1^4 \leq \|\varphi\|_{\frac{1}{2}} \|\varphi\|_{\frac{3}{2}}^3 \leq \|\varphi\|_{\frac{1}{2}} \|\varphi\|_2^3.$$

Interpolation in Orlicz classes

Theorem (Slavin, S., Vasyunin).
Let

$$\mathbf{B}(x_1,x_2)=\inf_{\varphi\ge 0}\{\langle\log(1+\varphi)\rangle_Q : \langle\varphi\rangle_Q=x_1; \langle(1+\varphi)\log(1+\varphi)\rangle_Q=x_2\}.$$

Then

$$\mathbf{B}(x_1,x_2)=\frac{x_2}{1+w},$$

where w *is the unique solution of the equation*

$$\frac{(1+w)\log(1+w)}{w}=\frac{x_2}{x_1}.$$

in the domain $(1,\infty)$.