AN EXTENSION OF THE CÓRDOBA-FEFFERMAN THEOREM ON THE EQUIVALENCE BETWEEN THE BOUNDEDNESS OF CERTAIN CLASSES OF MAXIMAL AND MULTIPLIER OPERATORS

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Abstract. The Córdoba-Fefferman Theorem involving the equivalence between boundedness properties of certain classes of maximal and multiplier operators is extended utilizing the recent work of Bateman on directional maximal operators as well as the work of Hagelstein and Stokolos on geometric maximal operators associated to homothecy invariant bases of convex sets satisfying Tauberian conditions.

It is well known that maximal and multiplier operators in harmonic analysis are fundamentally related. For example, the weak type bounds of the Hardy-Littlewood maximal operator on $\mathbb{R}^1$ are closely connected to the L^p bounds of the Hilbert transform for $1 < p < \infty$ (see, for instance, Chapter II of [5].) However, the interconnections between maximal and multiplier operators are still not completely understood, especially in higher dimensional settings. That being said, significant progress on this issue was made in the mid-1970's with the results of A. Córdoba and R. Fefferman in the context of a specific but useful class of maximal and multiplier operators [3]. Somewhat surprisingly, recent work on geometric maximal operators due to Bateman, Katz, and the authors ([1], [2], [4]) has enabled a substantial strengthening of Córdoba and Fefferman's results. The purpose of this note is to show how this recent work in the theory of geometric maximal operators may be used to extend the results of Córdoba and Fefferman, giving us an improved understanding of the interconnections between boundedness properties of maximal and multiplier operators in Fourier analysis.

We now recall the result of Córdoba and Fefferman found in [3]. Let $\theta_1 > \theta_2 > \theta_3 > \ldots$ be a decreasing sequence of angles between 0 and $\pi/2$. Let the geometric maximal operator M_θ be defined by

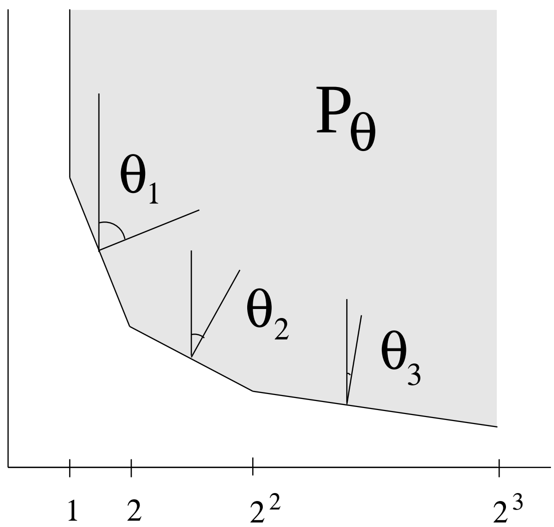
$$M_\theta f(x) = \sup_{x \in R} \frac{1}{|R|} \int_R |f(y)| \; dy$$

where the supremum is over the collection of rectangles in the plane of arbitrary eccentricity oriented in one of the directions θ_i. Associated to M_θ

is the multiplier operator T_θ given by

$$\widehat{T_\theta f}(\xi) = \chi_{P_\theta}(\xi) \cdot \hat{f}(\xi) \ ,$$

where P_θ is the subset of $\mathbb{R}^2$ as indicated below.



Córdoba and Fefferman proved the following:

Theorem 1. [3] *Let M_θ and T_θ be as indicated above.*

a) If M_θ is bounded on $L^p(\mathbb{R}^2)$, then T_θ is bounded on $L^q(\mathbb{R}^2)$ where $q = \frac{2p}{p-1}$.

b) If T_θ is bounded on $L^p(\mathbb{R}^2)$ for some $p > 2$ and M_θ satisfies the Tauberian condition

$$\left|\left\{x : M_\theta \chi_E(x) > \frac{1}{2}\right\}\right| \leq C\,|E|\,,$$

then M_θ is of weak type $\left(\left(\frac{p}{2}\right)', \left(\frac{p}{2}\right)'\right)$.

Two recent and seemingly unrelated results regarding geometric maximal operators will enable us to strengthen the above result of Córdoba and Fefferman. The first is from the work of Hagelstein and Stokolos on geometric maximal operators satisfying Tauberian conditions.

Theorem 2. [4] *Let $\mathcal{B}$ be a homothecy invariant collection of convex sets in $\mathbb{R}^2$. Define the maximal operator $M_{\mathcal{B}}$ by*

$$M_{\mathcal{B}} f(x) = \sup_{x \in R \in \mathcal{B}} \frac{1}{|R|} \int_R |f| \ .$$

Suppose for some $0 < \alpha < 1$ *there exists a positive finite constant* C_α *such that*

$$|\{x : M_{\mathcal{B}}\chi_E(x) > \alpha\}| \leq C_\alpha \, |E|$$

holds for every measurable set E *in* $\mathbb{R}^2$*. Then* $M_{\mathcal{B}}$ *is bounded on* $L^p(\mathbb{R}^2)$ *for sufficiently large* p*. In particular, there exists* $p_\alpha < \infty$ *depending only on* α*, and* C_α *such that* $M_{\mathcal{B}}$ *is bounded on* $L^p\left(\mathbb{R}^2\right)$ *for all* $p > p_\alpha$*.*

The result of Bateman of interest here (see also the related paper [2]) is the following.

Theorem 3. [1] *Let* Ω *be a set of directions in* $\mathbb{R}^2$*, and let* M_Ω *be the maximal operator associated to all rectangles oriented in those directions. If* M_Ω *is bounded on* $L^q(\mathbb{R}^2)$ *for some* $1 < q < \infty$*, then* M_Ω *is bounded on* $L^p(\mathbb{R}^2)$*for all* $1 < p < \infty$*.*

These three theorems may be combined to yield the following stronger version of Theorem 1 via a surprisingly short and direct proof.

Theorem 4. *Let* M_θ *and* T_θ *be as indicated above.*

a) If M_θ *is bounded on* $L^p(\mathbb{R}^2)$ *for some* $1 < p < \infty$*, then* M_θ *and* T_θ *are bounded on* $L^q(\mathbb{R}^2)$ *for all* $1 < q < \infty$*.*

b) If M_θ *satisfies the Tauberian condition*

$$\left|\left\{x : M_\theta \chi_E(x) > \frac{1}{2}\right\}\right| \leq C \, |E| \, ,$$

then M_θ *and* T_θ *are bounded on* $L^q(\mathbb{R}^2)$ *for all* $1 < q < \infty$*.*

Proof. a) M_θ is clearly a directional maximal operator of the type considered in Theorem 3. As by hypothesis it is bounded on $L^p(\mathbb{R}^2)$ for some $1 < p < \infty$ we see by Theorem 3 that M_θ is bounded on L^q for all $1 < q < \infty$. By Theorem 1 we then see T_θ is bounded on $L^p(\mathbb{R}^2)$ for $2 \leq p < \infty$. By duality we then see T_θ is bounded on $L^q(\mathbb{R}^2)$ for $1 < q < \infty$.

b) We are given that M_θ satisfies a Tauberian condition with respect to 1/2. By Theorem 2 we then see that M_θ must be bounded on $L^p(\mathbb{R}^2)$ for sufficiently large p. Applying part (a) we then achieve the desired result. □

By Theorem 4, we see that the L^p boundedness condition on T_θ in part (b) of Theorem 1 is rendered unnecessary - that in fact the desired conclusion follows just from the (previously considered weak) Tauberian condition on M_θ. The amount of information that can be gleaned just from the L^p

boundedness of T_θ remains unclear, however, and suggests the following problem certainly worthy of subsequent research:

Problem. *Let M_θ and T_θ be as indicated above. Suppose T_θ is bounded on $L^p(\mathbb{R}^2)$ for some $p > 2$. Must T_θ and M_θ be bounded on $L^q(\mathbb{R}^2)$ for all $1 < q < \infty$?*

References

[1] M. Bateman, Kakeya sets and directional maximal operators in the plane, arXiv: math/0703559v1.

[2] M. Bateman, N. H. Katz, Kakeya sets in Cantor directions, Math. Res. Lett. 15 (2008), no. 1, 73–81.

[3] A. Córdoba, R. Fefferman, On the equivalence between the boundedness of certain classes of maximal and multiplier operators in Fourier analysis, Proc. Nat. Acad. Sci. U.S.A. 74 (1977), no. 2, 423–425.

[4] P. Hagelstein, A. Stokolos, Tauberian conditions for geometric maximal operators, (to appear in Trans. A.M.S.)

[5] E. M. Stein, Singular Integrals and Differentiability Properties of Functions, Princeton University Press, 1970.

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