

# PHY 475/375

## Lecture 8

(April 18, 2012)

### A Universe of Many Components

So far, we have written down three equations that relate the scale factor  $a(t)$ , the energy density  $\varepsilon(t)$ , and the pressure  $P(t)$  in a homogenous and isotropic universe. To summarize, they are

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2} \varepsilon - \frac{\kappa c^2}{R_0^2} \frac{1}{a^2} \quad (5.1)$$

$$\dot{\varepsilon} + \frac{3\dot{a}}{a}(\varepsilon + P) = 0 \quad (5.2)$$

$$P = w\varepsilon \quad (5.3)$$

In reality, our Universe contains different components. The dimensionless equation of state parameters ( $w$ ) for these components, as we have learned, each have special values.

- The Universe contains non-relativistic matter, so it has components with  $w = 0$ .
- The Universe contains radiation, so it has components with  $w = 1/3$ .
- The Universe may contain a cosmological constant with  $w = -1$ .
- The Universe may contain even more exotic components with different values of  $w$ .

Fortunately, the energy density and pressure for the different components of the Universe are additive.

- We may write the total energy density  $\varepsilon$  as the sum of the energy density of the different components:

$$\varepsilon = \sum_w \varepsilon_w \quad (5.4)$$

where  $\varepsilon_w$  is the energy density of the component with equation-of-state parameter  $w$ .

- Meanwhile, the total pressure  $P$  is the sum of the pressures of the different components:

$$P = \sum_w P_w = \sum_w w\varepsilon_w \quad (5.5)$$

where we've used  $P = w\varepsilon$  from equation (5.3).

Since the energy densities and pressures add as above, the fluid equation (5.2) must hold separately for each component, as long as there is no interaction between the different components. Hence, the component with equation-of-state parameter  $w$  obeys the equation:

$$\dot{\varepsilon}_w + \frac{3\dot{a}}{a}(\varepsilon_w + P_w) = 0 \quad (5.6)$$

Since  $P_w = w\varepsilon_w$ , equation (5.6) can be written as

$$\dot{\varepsilon}_w + \frac{3\dot{a}}{a}(1+w)\varepsilon_w = 0 \quad (5.7)$$

Rearranging terms, we get

$$\frac{d\varepsilon_w}{dt} = -3(1+w) \left( \frac{1}{a} \frac{da}{dt} \right) \varepsilon_w$$

Multiplying both sides by  $1/\varepsilon_w$ , and writing terms like  $(1/x) dx/dt = d(\ln x)/dt$ , we get

$$\frac{d}{dt}(\ln \varepsilon_w) = -3(1+w) \frac{d}{dt}(\ln a)$$

Assuming  $w$  is constant and integrating, we get

$$\ln \left( \frac{\varepsilon_w}{\varepsilon_{w,0}} \right) = -3(1+w) \ln \left( \frac{a}{a_0} \right)$$

where  $\varepsilon_{w,0}$  is the energy density of the component with equation-of-state parameter  $w$  at the present epoch. Using the usual normalization that the scale factor at the present epoch,  $a_0 = 1$ , we get from this equation that

$$\varepsilon_w(a) = \varepsilon_{w,0} a^{-3(1+w)} \quad (5.9)$$

- Since  $w = 0$  for non-relativistic matter (subscripted with  $m$  below), equation (5.9) gives

$$\varepsilon_m(a) = \varepsilon_{m,0} a^{-3(1+0)}$$

so that

$$\varepsilon_m(a) = \frac{\varepsilon_{m,0}}{a^3} \quad (5.10)$$

- Meanwhile, since  $w = 1/3$  for radiation (subscripted with  $r$  below), we obtain from equation (5.9) that

$$\varepsilon_r(a) = \varepsilon_{r,0} a^{-3(1+1/3)}$$

so that

$$\varepsilon_r(a) = \frac{\varepsilon_{r,0}}{a^4} \quad (5.11)$$

We may ask why we get this difference between matter and radiation. The energy density of either component may be written in the form  $\varepsilon = nE$ , where  $n$  is the number density of particles, and  $E$  is the mean energy per particle. For both relativistic and non-relativistic particles, the number density has the dependence  $n \propto a^{-3}$  as the universe expands, assuming that particles are neither created nor destroyed.

- For non-relativistic particles, the energy is contributed solely by the rest mass of the particles ( $E = mc^2$ ) and remains constant as the universe expands. So, for non-relativistic matter,  $\varepsilon_m = nE = n(mc^2) \propto a^{-3}$ .
- On the other hand, the wavelength of radiation  $\lambda \propto a$ , as the universe expands, so the energy of radiation ( $E = hc/\lambda$ ) has the dependence  $E \propto a^{-1}$ . Therefore, for photons and other massless particles,  $\varepsilon_r = nE = n(hc/\lambda) \propto a^{-3} a^{-1} \propto a^{-4}$ .

Note that the above explanation required the assumption that photons are neither created nor destroyed. Such an assumption is, of course, not quite true, since photons are continuously being created. However, the number of Cosmic Microwave Background (CMB) photons is so large that the energy density of the CMB is larger than the energy density of all the photons emitted by all the stars in the history of the universe.

- Recall from an earlier lecture that the energy density of the CMB in the current epoch (with  $T_0 = 2.725$  K) is

$$\varepsilon_{\text{CMB}, 0} = \alpha T_0^4 = 4.17 \times 10^{-14} \text{ J m}^{-3} = 0.260 \text{ MeV m}^{-3} \quad (5.12)$$

- Meanwhile, the luminosity density of galaxies in the current epoch is

$$nL \approx 2 \times 10^8 L_\odot \text{ Mpc}^{-3} \approx 2.6 \times 10^{-33} \text{ watts m}^{-3} \quad (5.14)$$

For a crude estimate, let us assume that galaxies have been emitting light at this rate for the entire age of the universe,  $t_0 \approx H_0^{-1} \approx 14 \text{ Gyr} \approx 4.4 \times 10^{17} \text{ s}$ . This gives an energy density in starlight of

$$\begin{aligned} \varepsilon_{\text{starlight}, 0} &\approx nLt_0 \approx \left(2.6 \times 10^{-33} \text{ watts m}^{-3}\right) \left(4.4 \times 10^{17} \text{ s}\right) \\ &\approx 1 \times 10^{-15} \text{ J m}^{-3} \approx 0.007 \text{ MeV m}^{-3} \end{aligned} \quad (5.15)$$

Since

$$\frac{\varepsilon_{\text{starlight}, 0}}{\varepsilon_{\text{CMB}, 0}} \approx \frac{0.007 \text{ MeV m}^{-3}}{0.260 \text{ MeV m}^{-3}} = 0.03$$

the average energy density of starlight is currently only  $\sim 3\%$  of the energy density of the CMB.

While the above is a rough estimate, more refined estimates using measurements of background radiation from UV to the near-infrared, which includes both direct starlight and starlight absorbed and reradiated by dust, yield the larger value  $\varepsilon_{\text{starlight}, 0}/\varepsilon_{\text{CMB}, 0} \approx 0.1$ . Moreover, the ratio of starlight density to CM density was even smaller in the past.

Therefore, it is a reasonable approximation to ignore non-CMB photons when computing the mean energy density of photons in the universe.

Now, while it is true the energy density of CMB photons may be much higher than that of starlight, it is also the case that the energy density of CMB photons is small compared to the critical energy density. If we write the dimensionless density parameter defined in equation (4.28) in an earlier lecture, for CMB photons it will be equal to

$$\Omega_{\text{CMB}, 0} \equiv \frac{\varepsilon_{\text{CMB}, 0}}{\varepsilon_{c,0}} = \frac{0.260 \text{ MeV m}^{-3}}{5200 \text{ MeV m}^{-3}} = 5.0 \times 10^{-5} \quad (5.13)$$

Therefore, we do need to ask ourselves if there are any more contributions to the energy density in radiation, especially since equation (5.9) tells us that different components would dominate the universe at different times.

In fact, there is another contributor to the energy density in radiation, and it comes from primordial neutrinos. Recall that the CMB is a relic of the time when the Universe was hot and dense enough to be opaque to photons (as we discussed in an earlier lecture). In fact, if we extrapolated further back in time, we would reach an epoch when the universe was hot and dense enough to be opaque to neutrinos. As a consequence, there should be a Cosmic Neutrino Background today, although none has yet been detected because our detectors can't go low enough in energy, and can only detect neutrinos with  $E > 0.1$  MeV, whereas the mean energy per neutrino from this background is expected to be  $\sim 5 \times 10^{-4}$  eV/ $a$ . Still, we can calculate theoretically what the contribution of this neutrino background should be to the energy density in radiation.

Detailed calculations indicate that the density parameter of this cosmic neutrino background, taking into account all three flavors of neutrino, should be

$$\Omega_\nu = 0.681 \Omega_{\text{CMB}} \quad (5.17)$$

The density parameter for radiation in the current epoch is then equal to

$$\begin{aligned} \Omega_{r,0} &= \Omega_{\text{CMB},0} + \Omega_{\nu,0} \\ &= 5 \times 10^{-5} + 0.681 \left( 5 \times 10^{-5} \right) \\ \Rightarrow \Omega_{r,0} &= 8.4 \times 10^{-5} \end{aligned} \quad (5.19)$$

In a similar manner, we could estimate the density parameter for non-relativistic matter and the cosmological constant, and then find the relative contributions of each component. Unfortunately, the total energy density of non-relativistic matter is not so easy to predict, neither is the energy density of the cosmological constant. In later lectures, we will discuss how the available evidence favors a universe in which the density parameter for non-relativistic matter is currently  $\Omega_{m,0} \sim 0.3$ , while the density parameter for the cosmological constant may currently be  $\Omega_{\Lambda,0} \sim 0.7$ .

Therefore, when we want to use a model which matches the observed properties of the real Universe, we will use a so-called “Benchmark model” with the following parameters.

- The model has  $\Omega_{r,0} = 8.4 \times 10^{-5}$  in radiation.
- The model has  $\Omega_{m,0} = 0.3$  in non relativistic matter.
- The model has  $\Omega_{\Lambda,0} = 1 - \Omega_{m,0} \approx 0.7$  in a cosmological constant.

Note that your text predates the time around which the word “dark energy” came into use as the predominant term, so a more updated version might be to call this last component as dark energy (the difference being that the cosmological constant could be the dark energy, or that the dark energy might be something with a different value of  $w$ , as discussed later in this lecture).

For a universe containing multiple components, different components will dominate the global dynamics at different times, as revealed by equation (5.9):

$$\varepsilon_w(a) = \varepsilon_{w,0} a^{-3(1+w)}$$

- Equation (5.9) tells us that in the limit  $a \rightarrow 0$ , the component with the largest value of  $w$  is dominant. Therefore, radiation ( $w = 1/3$ ) would have been dominant during the early stages of our Universe.
- At some point in the history of the Universe, the radiation dominance would have given way to dominance by non-relativistic matter. In fact, we can calculate when this would have happened, using the ratio of energy density in non-relativistic matter and energy density in radiation from equations (5.10) and (5.11) respectively:

$$\frac{\varepsilon_m(a)}{\varepsilon_r(a)} = \frac{\varepsilon_{m,0}/a^3}{\varepsilon_{r,0}/a^4} = \left( \frac{\varepsilon_{m,0}}{\varepsilon_{r,0}} \right) a \quad (5.24)$$

This ratio would have been 1 when radiation-matter equality manifested itself, so

$$1 = \left( \frac{\varepsilon_{m,0}}{\varepsilon_{r,0}} \right) a_{(r=m)}$$

Thus, the scale factor at the time of radiation-matter equality was

$$a_{(r=m)} = \left( \frac{\varepsilon_{m,0}}{\varepsilon_{r,0}} \right)^{-1} = \left( \frac{\varepsilon_{m,0}/\varepsilon_{c,0}}{\varepsilon_{r,0}/\varepsilon_{c,0}} \right)^{-1} \quad (5.25.a)$$

where  $\varepsilon_{c,0}$  is the critical energy density at the current epoch.

Since  $\Omega(a) = \varepsilon(a)/\varepsilon_c(a)$ , we get from equation (5.25.a) that

$$a_{(r=m)} = \left( \frac{\Omega_{m,0}}{\Omega_{r,0}} \right)^{-1} \approx \left( \frac{0.3}{8.4 \times 10^{-5}} \right)^{-1} = \frac{1}{3600} = 2.8 \times 10^{-4} \quad (5.25)$$

Therefore, matter would have started to dominate over radiation when the scale factor was  $1/3600$ .

- Equation (5.9) also tells us that as  $a \rightarrow \infty$ , the component with the smallest value of  $w$  would be dominant. If the dark energy is caused by a cosmological constant with  $w = -1$ , then we are currently dominated by the cosmological constant; we can see this easily by calculating the ratio of the energy density in  $\Lambda$  to the energy density in matter in the current epoch:

$$\frac{\varepsilon_{\Lambda,0}}{\varepsilon_{m,0}} = \frac{\varepsilon_{\Lambda,0}/\varepsilon_{c,0}}{\varepsilon_{m,0}/\varepsilon_{c,0}} = \frac{\Omega_{\Lambda,0}}{\Omega_{m,0}} \approx \frac{0.7}{0.3} = 2.3 \quad (5.20)$$

Clearly, therefore, we are dominated by the dark energy at the current epoch; note that we've assumed here that the dark energy is contributed by the cosmological constant.

In fact, we can calculate when the transition from matter-dominance to  $\Lambda$ -dominance occurred. To do this, first realize using equation (5.9) that with  $w = -1$  for the cosmological constant, we must have

$$\varepsilon_{\Lambda}(a) = \varepsilon_{\Lambda,0} a^{-3(1+w)} = \varepsilon_{\Lambda,0} a^{-3(1+\{-1\})} = \varepsilon_{\Lambda,0} a^{-3(0)} = \varepsilon_{\Lambda,0} a^0$$

That is,  $\varepsilon_{\Lambda}(a) = \varepsilon_{\Lambda,0}$ , meaning that the energy density in  $\Lambda$  remains constant with time.

Then, using the ratio of densities

$$\frac{\varepsilon_{\Lambda}(a)}{\varepsilon_m(a)} = \frac{\varepsilon_{\Lambda,0}}{\varepsilon_{m,0}/a^3} = \left(\frac{\varepsilon_{\Lambda,0}}{\varepsilon_{m,0}}\right) a^3 \quad (5.21)$$

This ratio would have been 1 when the equality between the energy density in matter and that in the cosmological constant manifested itself, so

$$1 = \left(\frac{\varepsilon_{\Lambda,0}}{\varepsilon_{m,0}}\right) a_{(m=\Lambda)}^3$$

Thus, the scale factor at the time of matter- $\Lambda$  equality was

$$a_{(m=\Lambda)} = \left(\frac{\varepsilon_{\Lambda,0}}{\varepsilon_{m,0}}\right)^{-1/3} = \left(\frac{\varepsilon_{\Lambda,0}/\varepsilon_{c,0}}{\varepsilon_{m,0}/\varepsilon_{c,0}}\right)^{-1/3} = \left(\frac{\Omega_{\Lambda,0}}{\Omega_{m,0}}\right)^{-1/3} = \left(\frac{0.7}{0.3}\right)^{-1/3} \approx 0.75 \quad (5.22)$$

Therefore, if the dark energy is comprised of a cosmological constant, it would have started to dominate over matter when the scale factor was 0.75. Since  $a(t_0) = 1$  at the present time, this means that we have only recently entered a period when the cosmological constant is the dominant component.

Note that in all of the above discussion, we have used the scale factor  $a$  to refer to unique periods in the history of the universe, much like a stand-in for the cosmic time itself. We shall continue to do so, more so because there is a simple relation between scale factor and redshift. Recall that we wrote in an earlier lecture:

$$1 + z = \frac{a(t_0)}{a(t_e)}$$

and since  $a(t_0) = 1$  in the present epoch, this can be written as

$$1 + z = \frac{1}{a}$$

where we have suppressed writing  $t$  for convenience. So, even redshift is often used as a surrogate for time, especially since  $z$  is a measured quantity that doesn't depend on any implicit assumptions about the universe.

For example, since radiation-matter equality took place when the scale factor  $a$  was  $1/3600$ , as we discussed above, we will say that radiation-matter equality took place at a redshift  $z_{(r=m)} \approx 3600$ . Of course, it really should be  $(1/a) - 1$ , but for such large numbers, that's where the significant figures would put it).

In the next class, we will look at simplified models of the universe consisting of only one component. While this is unrealistic, such an exercise yields useful insight into the actual Universe.