PHY 375
Homework 4 solutions
(Due by beginning of class on Wednesday, May 16, 2012)

1. The surface brightness Σ of an astronomical object is defined as its observed flux divided by its observed angular area; that is, $\Sigma \propto f/(\delta\theta)^2$.

(a) For a class of objects which are both standard candles and standard yardsticks, what is Σ as a function of redshift?

To save time, I've scanned in some of my handwritten solutions for this homework. I've generally avoided this, since after years of escaping from handwriting except on the blackboard, my skills at paper-based handwriting are certainly far past their prime. So, if something is not legible, please feel free to ask for clarifications.

Solution:

$$\Sigma \propto \frac{f}{(d\theta)^2} \equiv \frac{L/4\pi d_L^2}{(\ell/d_A)^2}$$

eq (7.21), Lecture 12 using flux of a standard candle to define d_L, the luminosity distance

eq. (7.33), Lecture 12 definition of angular diameter distance d_A for a yardstick of proper length ℓ

But, from eq. (7.37), lecture 12

$$d_A = \frac{d_L}{(1+z)^2}$$

So

$$\Sigma \propto \frac{L}{4\pi\ell^2}\frac{d_A^2}{d_L^2} = \frac{L}{4\pi\ell^2}\frac{\cancel{d_L^2}/(1+z)^4}{\cancel{d_L^2}}$$

finally, therefore,

$$\boxed{\Sigma \propto \frac{L}{4\pi\ell^2}\frac{1}{(1+z)^4} \quad \text{i.e.} \quad \Sigma \propto \frac{1}{(1+z)^4}}$$

(b) Would observing the surface brightness of this class of objects be a useful way of determining the deceleration parameter q_0? Explain clearly why or why not.

Solution:

Since Σ does not contain d_A or d_L separately, but only their ratio, it is not a useful way of determining q_0.

2. When a photon passes an object of mass M at an impact parameter b (see Figure 8.5 in your text), it will be deflected by an angle $\alpha = 4GM/c^2 b$ (equation 8.48).

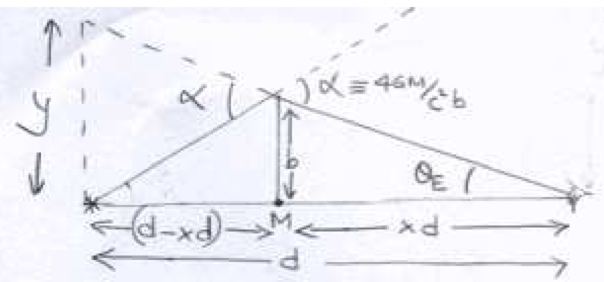
   Suppose now that an object of mass M is acting as a gravitational lens. To keep things simple, assume that this lens is exactly along the line of sight between the observer and the lensed object. The distance from observer to the lensed object is d, and the distance from observer to the lens is xd, where $0 < x < 1$.

(a) Show that the angular radius of the Einstein ring for this configuration is given by

$$\theta_E = \left(\frac{4GM}{c^2 d}\,\frac{1-x}{x}\right)^{1/2}$$

Remember that due to the large distances involved, all angles are small, and consequently, $\sin\theta \approx \tan\theta \approx \theta$, provided θ is expressed in radians.

Solution:



From the figure,

$$\frac{b}{xd} = \theta_E \text{ (rad)}, \quad \& \quad \frac{y}{d} = \theta_E \text{ (rad)}$$

Since all angles are small, $\sin\theta \approx \tan\theta \approx \theta$

Also, using $s = r\theta$

we get $y = (d-xd)\alpha$ → again, with the large distances involved, arc $s \equiv$ straight line

$$\Rightarrow \theta_E = \frac{y}{d} = \frac{(d-xd)\alpha}{d} = \frac{4GM}{c^2 b}(1-x) = \frac{4GM}{c^2 \underbrace{xd\,\theta_E}_{=b}}(1-x)$$

$$\Rightarrow \theta_E^2 = \frac{4GM}{c^2 d}\,\frac{1-x}{x} \Rightarrow \theta_E = \left(\frac{4GM}{c^2 d}\,\frac{1-x}{x}\right)^{1/2}$$

(b) For what value of the fraction x would you get the best Einstein ring (i.e., the Einstein ring with the largest angular radius). Show your calculations/derivations clearly.

Solution:

If you do $d\theta_E/dx = 0$, or run through values of $(1-x)/x$, you will find that the largest θ_E will occur for $x \to 0$. While this makes perfect sense mathematically, it is problematic on physical grounds, since even though our derived result is strictly true only for a point source lens, even a point source will start to exhibit extended structure if it approaches so close to your viewing location (not to mention that you're about to get rammed by a solar mass object!). Therefore, it is well worth taking note of the fact that the Einstein ring would only be visible if $\theta_E > \theta_R$, where θ_R is the angle subtended by the extended source at the object. So, since $(1-x)/x = 1$ at $x = 1/2$ (i.e., when the lens is halfway between the lensed object and observer), and increases as we get closer to the observer, the best Einstein ring would be obtained somewhere between $x = 1/2$ and $x = 0$ at the position where $\theta_E \approx \theta_R$.

(c) Find the angular radius θ_E of the Einstein ring (in arcsec) for a microlensing event in which the lens is a *MACHO* of mass $0.3M_\odot$ in the Galactic halo at a distance of 10 kpc from the Earth, and the lensed object is a star in the LMC at a distance of 50 kpc from the Earth.

Note that $1M_\odot = 1.99 \times 10^{30}$ kg, and π radians $\equiv 180 \times 3600$ arcsec.

Solution: This is a straightforward substitution for θ_E with $x = 10\text{ kpc}/50\text{ kpc} = 1/5$. So

$$\theta_E = \left[\frac{4\left(6.67 \times 10^{-11}\text{ m}^3\text{ kg}^{-1}\text{ s}^{-2}\right)\left(0.3M_\odot\left\{1.99 \times 10^{30}\text{ kg}/M_\odot\right\}\right)}{\left(3 \times 10^8\text{ m s}^{-1}\right)^2\left(50,000\text{ pc}\left\{3.1 \times 10^{16}\text{ m/pc}\right\}\right)}\;\frac{1-1/5}{1/5}\right]^{1/2}$$

from which we get

$$\theta_E = 2.1 \times 10^{-9}\text{ rad} = \boxed{\mathbf{4.4 \times 10^{-4}\ arcsec}}$$

As expected, this is very small. Recall that in microlensing events, the Einstein ring is too small in radius to be resolved.

(d) If the timescale for a microlensing event is the time it takes for a *MACHO* to travel through an angular distance equal to the Einstein radius θ_E as seen from Earth, calculate the time (in days) for the microlensing event described in part (c), if the *MACHO* is moving at 200 km s^{-1}.

Solution: This is just the time taken to move the equivalent of an angular distance of 2.1×10^{-9} rad at 10 kpc for an object moving at 200 km/s, so

$$t = \frac{\left(2.1 \times 10^{-9}\text{ rad}\right)\ 10,000\text{ pc}\left(3.1 \times 10^{16}\text{ m/pc}\right)}{200,000\text{ m s}^{-1}} = 3.3 \times 10^6\text{ s} \approx \boxed{\mathbf{38\ days}}$$

3. A spatially flat universe contains a single component with equation of state parameter w. Consider a standard candle of luminosity L located at redshift z in this universe. Show that the observed flux from this standard candle is

$$f(z) = \frac{L\,(1+3w)^2}{16\pi c^2}\,\frac{H_0^2}{(1+z)^2}\left[1 - \frac{1}{(1+z)^{(1+3w)/2}}\right]^{-2}$$

Solution:

The quantity in square brackets looks like part of the expression for $d_p(t_0)$ from equation (5.54) in Lecture 9, and since we know from equation (7.21) in Lecture 12 that $f = L/4\pi d_L^2$, we need a relation between d_L and $d_p(t_0)$.

We can get such a relation from equation (7.28) in Lecture 12:

$$d_L = S_\kappa(r)\Big(1+z\Big) = r\Big(1+z\Big) = d_p(t_0)\,\Big(1+z\Big)$$

where we have used $S_\kappa(r) = r$ for a flat universe, and $d_p(t_0) = r$ from equation (5.33) in Lecture 9.

So, we get

$$f = \frac{L}{4\pi d_L^2} = \frac{L}{4\pi d_p(t_0)^2(1+z)^2}$$

Now, for a spatially flat universe with one component (having equation-of-state parameter w), equation (5.54) in Lecture 9 gives

$$d_p(t_0) = \frac{c}{H_0}\,\frac{2}{1+3w}\left[1 - \frac{1}{(1+z)^{(1+3w)/2}}\right]$$

So,

$$f(z) = \frac{L}{4\pi\left\{\left(\frac{c}{H_0}\right)^2\left(\frac{2}{1+3w}\right)^2\left[1 - \frac{1}{(1+z)^{(1+3w)/2}}\right]^2\right\}(1+z)^2}$$

or

$$f(z) = \frac{L\,H_0^2\,(1+3w)^2}{4\pi c^2(4)(1+z)^2}\left[1 - \frac{1}{(1+z)^{(1+3w)/2}}\right]^{-2}$$

Rearranging terms, we obtain the required result

$$f(z) = \frac{L\,(1+3w)^2}{16\pi c^2}\,\frac{H_0^2}{(1+z)^2}\left[1 - \frac{1}{(1+z)^{(1+3w)/2}}\right]^{-2}$$