PHY 375

Homework 3 solutions

(Due by beginning of class on Wednesday, April 25, 2012)

1. The cosmological constant has come under renewed scrutiny in recent years (with a different value from Einstein's, of course), because it may be a contributor to the dark energy that is responsible for the acceleration of the expansion of the Universe.

(a) Calculate the energy density of the cosmological constant in the current epoch, assuming $\Omega_\Lambda = 0.7$ and $H_0 = 70$ km s^{-1} Mpc^{-1}.

Solution: Since $\Omega_\Lambda = \varepsilon_{\Lambda,0}/\varepsilon_{c,0}$, we get

$$\varepsilon_{\Lambda,0} = \Omega_\Lambda\, \varepsilon_{c,0} = 0.7\left(8.3 \times 10^{-10}\ \text{J m}^{-3}\right) = 5.8 \times 10^{-10}\ \text{J m}^{-3}$$

or if we want this in MeV m^{-3}

$$\varepsilon_{\Lambda,0} = \Omega_\Lambda\, \varepsilon_{c,0} = 0.7\left(5200\ \text{MeV m}^{-3}\right) = 3640\ \text{MeV m}^{-3}$$

Therefore the energy density of the cosmological constant in the current epoch is

$$\boldsymbol{\varepsilon_{\Lambda,0} = 5.8 \times 10^{-10}\ \textbf{J m}^{-3} = 3640\ \textbf{MeV m}^{-3}}$$

(b) What is the total energy of the cosmological constant within a sphere 1 AU in radius?

Solution: The total energy with a sphere $r = 1$ AU in radius is then

$$E_{1\ \text{AU}} = \varepsilon_{\Lambda,0}\left[\frac{4\pi}{3} r^3\right] = 5.8 \times 10^{-10}\ \text{J m}^{-3}\left[\frac{4\pi}{3}\left(150 \times 10^9\ \text{m}\right)^3\right]$$

Therefore, the total energy within a sphere 1 AU in radius is equal to

$$E_{1\ \text{AU}} = \mathbf{8.2 \times 10^{24}\ J} \equiv= \mathbf{5.1 \times 10^{37}\ MeV}$$

(c) What is the rest energy of the Sun ($E_\odot = M_\odot\, c^2$)?

Solution: The rest energy of the Sun is equal to

$$E_\odot = M_\odot\, c^2 = \left(1.99 \times 10^{30}\ \text{kg}\right)\left(3 \times 10^8\ \text{m s}^{-1}\right)^2 = \mathbf{1.8 \times 10^{47}\ J}$$

(d) Comparing your answers above, do you expect the cosmological constant to have a significant effect on the motion of planets within the Solar System?

Solution: Since the energy in the solar neighborhood is dominated by that of the Sun, whose rest energy is many orders of magnitude larger than the energy of the cosmological constant in a sphere of radius 1 AU, we don't expect the cosmological constant to have any significant effect on the motion of planets within the Solar System.

2. Suppose the mass density of the universe was $\rho = 3 \times 10^{-27}$ kg m^{-3}.

(a) What would be the radius of curvature R_0 of Einstein's static universe?

Solution:

From equation (4.69) in Lecture 7, the radius of curvature of Einstein's static universe is

$$R_0 = \frac{c}{\Lambda^{1/2}}$$

where $\Lambda = 4\pi G\rho$.

Therefore, we get

$$R_0 = \frac{3 \times 10^8 \text{ m s}^{-1}}{\sqrt{4\pi\ (6.67 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2})(3 \times 10^{-27} \text{ kg m}^{-3})}}$$

so that the radius of curvature is

$$R_0 = \mathbf{1.9 \times 10^{26}\ m \equiv 6100\ Mpc}$$

(b) How long would it take a photon to circumnavigate such a universe?

Solution:

This is easy, since we just have a static universe. Moreover, the universe is positively curved.

So, all the photon needs to do is circumnavigate a sphere of radius R_0.

Therefore, the time taken is

$$t = \frac{\text{Circumference}}{c} = \frac{2\pi R_0}{c}$$

or

$$t = \frac{2\pi\ (1.9 \times 10^{26} \text{ m})}{3 \times 10^8 \text{ m s}^{-1}} = \mathbf{4 \times 10^{18}\ s \equiv 130\ Gyr}$$

Another interesting way to express this time is to put it in terms of the Hubble time $t_0 = H_0^{-1}$. We will get

$$t = 4 \times 10^{18} \text{ s} \left(\frac{70,000}{3.1 \times 10^{22}} \frac{\text{s}^{-1}}{H_0} \right) = \mathbf{9\ H_0^{-1}}$$

3. In a *flat universe* with $H_0 = 70$ km s^{-1} Mpc^{-1}, you observe a galaxy at a redshift $z = 7$. Carry out calculations to find the current proper distance to the galaxy, $d_p(t_0)$, in each of the following 3 cases. Also, carry out calculations to find the proper distance at the time the light was emitted, $d_p(t_e)$, again in each of the following 3 cases.

(a) Show your calculations for $d_p(t_0)$ and $d_p(t_e)$ if the universe contains *only radiation*?

Solution:

In a spatially flat universe, the proper distance at the time of observation and at the time of emission respectively is given by:

$$d_p(t_0) = \frac{c}{H_0}\frac{2}{1+3w}\left[1 - \frac{1}{(1+z)^{(1+3w)/2}}\right] \qquad \text{and} \qquad d_p(t_e) = \frac{d_p(t_0)}{1+z}$$

if $w \neq -1$.

Therefore, in a universe containing only radiation ($w = 1/3$), we get

$$d_p(t_0) = \frac{c}{H_0}\left[1 - \frac{1}{(1+z)}\right] = \frac{3 \times 10^8 \text{ m/s}}{70,000 \text{ m/s Mpc}^{-1}/(3.1 \times 10^{22} \text{ m/Mpc})}\left[1 - \frac{1}{(1+7)}\right]$$

so we get

$$d_p(t_0) = \frac{3 \times 10^8 \ (3.1 \times 10^{22})}{70,000}\left(\frac{7}{8}\right) = \mathbf{1.2 \times 10^{26} \ m \equiv 3750 \ Mpc}$$

and

$$d_p(t_e) = \frac{d_p(t_0)}{1+z} = \frac{3750}{1+7} = \mathbf{470 \ Mpc}$$

(b) Show your calculations if the universe contains *only matter*?

Solution:

Using the same expression as above, except with $w = 0$ for matter, we get

$$d_p(t_0) = \frac{2c}{H_0}\left[1 - \frac{1}{\sqrt{1+z}}\right] = \frac{(2)\ 3 \times 10^8 \text{ m/s}}{70,000 \text{ m/s Mpc}^{-1}/(3.1 \times 10^{22} \text{ m/Mpc})}\left[1 - \frac{1}{\sqrt{1+7}}\right]$$

so we get

$$d_p(t_0) = \frac{(2)\ 3 \times 10^8 \ (3.1 \times 10^{22})}{70,000}\left(\frac{1.828}{2.828}\right) = \mathbf{1.7 \times 10^{26} \ m \equiv 5500 \ Mpc}$$

and

$$d_p(t_e) = \frac{d_p(t_0)}{1+z} = \frac{5500}{1+7} = \mathbf{690 \ Mpc}$$

(c) Show your calculations if the universe contains *only a cosmological constant*?

Solution:

In a universe containing only a cosmological constant, the proper distance is given by equation (5.79) from Lecture 9:

$$d_p(t_0) = \frac{c}{H_0}\,z = \frac{3 \times 10^8 \text{ m/s}}{70,000 \text{ m/s Mpc}^{-1}/(3.1 \times 10^{22} \text{ m/Mpc})}\left(7\right)$$

so that

$$d_p(t_0) = \frac{3 \times 10^8\ (3.1 \times 10^{22})\ 7}{70,000} = \mathbf{9.3 \times 10^{26}\ m \equiv 30,000\ Mpc}$$

and

$$d_p(t_e) = \frac{d_p(t_0)}{1+z} = \frac{30,000}{1+7} = \mathbf{3750\ Mpc}$$

(d) Put all your answers for $d_p(t_0)$ and $d_p(t_e)$ for the three cases above in a table.

Solution:

	Nature of universe	$d_p(t_0)$ (Mpc)	$d_p(t_e)$ (Mpc)
(a)	Radiation only	3800	470
(b)	Matter only	5500	690
(c)	Cosmological Constant only	30,000	3800