DIRECTIONAL ERGODICITY AND MIXING FOR ACTIONS OF $\mathbb{Z}^d$

E. ARTHUR ROBINSON JR, JOSEPH ROSENBLATT, AND AYŞE A. ŞAHİN

1. Introduction

In this paper we define directional ergodicity and directional weak mixing for finite measure preserving $\mathbb{Z}^d$ and $\mathbb{R}^d$ actions and we study the structure of the set of directions for which an ergodic or weak mixing action fails to be directionally ergodic or weak mixing. Given a dynamical system defined by the action of a group G, it is natural to study the sub-dynamics of the action. In particular, one can ask what dynamical properties of the action of G are inherited by the group actions one obtains by restricting the original action to sub-groups of G. In the 1980's Milnor introduced the more general idea of *directional dynamics* for $\mathbb{Z}^d$ actions [6]. He defined the directional entropy of a $\mathbb{Z}^d$ action in all directions, including irrational ones. The study of directional entropy has been a productive line of research (see for example[7], [8], [10], [13]). In addition, the idea of defining directional dynamical properties more generally has led to other advances in dynamics, most notably expansive sub-dynamics introduced in [2].

There has been recent interest in directional recurrence properties of discrete group actions. In [5] Johnson and one of the authors investigated directional recurrence properties of infinite measure preserving actions of $\mathbb{Z}^d$. Their work has been generalized by Danilenko [3] who also investigated directional rigidity properties of infinite measure preserving actions of $\mathbb{Z}^d$ and of the Heisenberg group. There the author establishes a framework for studying these questions for actions of groups Γ that are lattices in simply connected nilpotent Lie groups.

In this paper we investigate the directional ergodicity and weak mixing properties of $\mathbb{Z}^d$ actions. There are multiple examples in the literature to suggest that mixing properties (spectral properties more generally) of a $\mathbb{Z}^d$ action are not necessarily inherited even by its sub-actions. We note in particular an example of Bergelson and Ward [1] from the late '90's (which we discuss in detail below) of a weak mixing $\mathbb{Z}^2$ action with no ergodic sub-actions, the example of Ferenczi and Kaminski that

Date: February 25, 2015.

is weak mixing and rigid, but all of whose one-dimensional sub-actions are infinite entropy Bernoulli actions [4], and Rudolph's example [11] of a rank one $\mathbb{Z}^2$ action with a Bernoulli one-dimensional sub-action. Other related work in this area for $\mathbb{Z}^d$ actions includes Ward's work on mixing of all orders in oriented cones [14] for algebraic actions of $\mathbb{Z}^d$, as well as Ryzhikov's result [12] that the generic $\mathbb{Z}^d$ action can be embedded only in an $\mathbb{R}^d$ action all of whose sub-actions are weak mixing. Directional ergodicity was investigated for continuous groups as well. Pugh and Shub [9] gave a spectral characterization of the failure of directional ergodicity along an e-dimensional direction of an ergodic $\mathbb{R}^d$ action, $e < d$. Our work here is most closely related to the work of Pugh and Shub and of Ryzhikov.

Let (X, μ) denote a non-atomic Lebesgue probability space and let $\{T^g\}_{g \in G}$ denote a measurable and measure preserving action of $G = \mathbb{Z}^d$ or $\mathbb{R}^d$ on X. We abbreviate the triple $\left(X, \mu, \{T^g\}_{\{g \in G\}}\right)$ by T in case $G = \mathbb{Z}^d$ and $\mathcal{T}$ if $G = \mathbb{R}^d$. Given $1 \le e \le d$, let $\mathbb{G}_{e,d}$ denote the Grassmanian manifold of e-dimensional planes in $\mathbb{R}^d$, and set $\mathbb{G}_d = \bigcup_e \mathbb{G}_{e,d}$. We define an *$e$-dimensional direction in $\mathbb{R}^d$ or $\mathbb{Z}^d$* to be an element $L \in \mathbb{G}_{e,d}$. We note that if $e = 1$ and $d = 2$ then a direction is specified by an angle $\theta \in [0, \pi)$.

In the existing work on directional dynamics there have been two approaches to defining directional properties for $\mathbb{Z}^d$ actions. One can define the directional property in a direction $L \in \mathbb{G}_{e,d}$, as in Milnor's definition of directional entropy, intrinsically using rational approximants $\vec{n}_i$ of L and studying the behavior of the collection $\{T^{\vec{n}_i}\}$. Alternatively, given a direction L, one can associate to T an $\mathbb{R}^e$ action in that direction by considering the unit suspension flow of T, restricted to the subgroup corresponding to the direction L. For directional entropy the two definitions are equivalent [8]. In the infinite dimensional case the two definitions are also equivalent for directional recurrence [5] but not for directional rigidity and the question for recurrence in the case of groups Γ as described above remains open [3].

Here we begin by defining directional ergodicity and weak mixing for a $\mathbb{Z}^d$ action via its unit suspension. We expand the work of Pugh and Shub to include a characterization of directional weak mixing of an $\mathbb{R}^d$ action in terms of its spectral measure. We then prove a structure theorem relating the maximal spectral types of a $\mathbb{Z}^d$ action and that of its unit suspension. This theorem allows us to give an intrinsic definition of directional ergodicity and weak mixing of the $\mathbb{Z}^d$ action in terms of its maximal spectral type, even in irrational directions. In a separate paper we address the question of defining directional weak mixing and ergodicity intrinsically in the $\mathbb{Z}^d$ action using the behavior of the action along approximants of the direction.

We address several additional questions. We consider the structure of the sets of directions $\mathcal{L} \subset \mathbb{G}_d$ that can fail to be ergodic or weak mixing for a $\mathbb{Z}^d$ action and give examples achieving certain possible sets of bad directions.

We also note that if a $\mathbb{Z}^d$ action T embeds in an $\mathbb{R}^d$ action $\mathcal{T}$, it would be possible to define directional ergodicity and weak mixing of T using the behavior of $\mathcal{T}$ along the corresponding subgroup of $\mathbb{R}^d$. We show that the unit suspension and embedding definitions always coincide for directional weak mixing, for any $\mathbb{Z}^d$ action. On the other hand, we show that an ergodicity assumption on T is both necessary and sufficient to preserve directional ergodicity between embeddings and suspensions.

Ryzhikov's work in [12], placed in the context we provide in this paper, shows that a generic $\mathbb{Z}^d$ action is weak mixing in all directions. Here we give a Fourier analytic proof, using the spectral characterizations that we provide for directional weak mixing. Using similar techniques we also provide an example of a rigid, weak mixing $\mathbb{Z}^d$ action that is weak mixing in every direction.

We now provide more formal statements of our results.

1.1. Defining directional properties.

A direction $L \in \mathbb{G}_{e,d}$ corresponds to a subgroup of $\mathbb{R}^d$. Given an $\mathbb{R}^d$ action $\mathcal{T}$, ergodicity and weak mixing in the direction L then have an intrinsic meaning in terms of the subgroup action $\mathcal{T}_L = (X, \mu, \{\mathcal{T}^{\vec{v}}\}_{\vec{v} \in L})$.

Definition 1.1. We say an $\mathbb{R}^d$ action $\mathcal{T}$ is ergodic (or weak mixing) in the direction $L \in \mathbb{G}_{e,d}$, if the restriction of $\mathcal{T}$ to the subgroup corresponding to L, $\mathcal{T}_L$ is ergodic (weak mixing).

As mentioned in the previous section, we wish to define ergodicity or weak mixing of a $\mathbb{Z}^d$ action T in a direction $L \in \mathbb{G}_{e,d}$ in terms of the behavior of its unit suspension along the corresponding subgroup of $\mathbb{R}^d$. We establish some notation necessary to define a unit suspension flow. Let λ denote Lebesgue measure on $[0,1)^d$, for $\vec{w} \in \mathbb{R}^d$ let $\lfloor \vec{w} \rfloor$ denote the vector in $\mathbb{Z}^d$ obtained by taking the floor of each component of $\vec{w}$, and let $\{\vec{w}\} = \vec{w} - \lfloor \vec{w} \rfloor$. The *unit suspension* of T is the $\mu \times \lambda$-preserving $\mathbb{R}^d$ action $\tilde{\mathcal{T}} = (X \times [0,1)^d, \mu \times \lambda, \{\tilde{\mathcal{T}}^{\vec{v}}\}_{\vec{v} \in \mathbb{R}^d})$ defined by

$$\tilde{\mathcal{T}}^{\vec{v}}(x, \vec{r}) = (T^{(\lfloor \vec{v} + \vec{r} \rfloor)} x, \{\vec{v} + \vec{r}\}).$$

Before we use the directional behavior of $\tilde{\mathcal{T}}$ to define directional properties of T we note that there are some obvious obstructions to directional ergodicity and weak mixing for a unit suspension $\mathbb{R}^d$ action introduced by functions that depend only on the torus and hence not relevant to the dynamics of T. In order to eliminate these obstructions we make some further definitions.

We let $\hat{G}$ denote the dual of the group G and we recall the following classical definition of the point spectrum of a G action T:

$$\mathrm{Spec}(T) = \{\gamma \in \hat{G} : \exists\, f \in L^2(X,\mu) \text{ with } f(T^g x) = \gamma(g)\cdot f(x)\ \forall g \in G\}.$$

An element $\gamma \in \mathrm{Spec}(T)$ is called an *eigenvector* of T and a function f defined as above is called an *eigenfunction* for T. If $G = \mathbb{Z}^d$, then for $\gamma \in \hat{G}$, $\gamma : \mathbb{Z}^d \to \mathbb{C}$, is given by $\gamma(\vec{n}) = e^{2\pi i \vec{\alpha}\cdot\vec{n}}$, for some $\vec{\alpha} \in \mathbb{R}^d$. But if $\vec{\alpha}' = \vec{\alpha} + \vec{n}$ for $\vec{n} \in \mathbb{Z}^d$, then the corresponding $\gamma' = \gamma$, so an element $\vec{\alpha} \in \mathbb{T}^d$ determines γ uniquely, and we identify $\hat{G} = \mathbb{T}^d$. Similarly, when $G = \mathbb{R}^d$ then $\gamma \in \hat{G}$, $\gamma : \mathbb{Z}^d \to \mathbb{C}$, is given by $\gamma(\vec{v}) = e^{2\pi i \vec{v}\cdot\vec{\omega}}$, for some $\vec{\omega} \in \mathbb{R}^d$, where now since $\vec{\omega}$ completely determines γ, we identify $\hat{G} = \mathbb{R}^d$.

Define the set $\mathcal{G} \subset L^2(X \times [0,1)^d, \mu \times \lambda)$ given by functions $f(x,\vec{r}) = G(\vec{r})$ for some $G \in L^2([0,1)^d, \lambda)$. Let $\mathcal{H} = \mathcal{G}^\perp$ so that $L^2(X \times [0,1)^d, \mu \times \lambda) = \mathcal{G} \oplus \mathcal{H}$. We say $\vec{\omega} \in \mathrm{Spec}_{\mathcal{H}}(\mathcal{T})$ if $\vec{\omega}$ is an eigenvalue for $\mathcal{T}$ with an eigenfunction $f \in \mathcal{H}$. Note that the constant functions belong to $\mathcal{G}$ and so are orthogonal to any $f \in \mathcal{H}$. We are now ready to define directional weak mixing and ergodicity for $\mathbb{Z}^d$ actions.

Definition 1.2. Let T be a measurable and measure preserving action of $\mathbb{Z}^d$ on a Lebesgue space (X,μ) with unit suspension $\tilde{\mathcal{T}}$. Let $L \in \mathbb{G}_{e,d}$ for $1 \le e \le d$.

(1) T is *ergodic in the direction* L if $\vec{0} \notin \mathrm{Spec}_{\mathcal{H}}(\tilde{\mathcal{T}}_L)$.
(2) T is *weakly mixing in the direction* L or has *continuous spectrum in the direction* L if $\mathrm{Spec}_{\mathcal{H}}(\tilde{\mathcal{T}}_L) = \varnothing$.

Finally, we establish notation for the ergodic and weak mixing directions of a $\mathbb{Z}^d$ action.

Definition 1.3. Let T be a measurable and measure preserving $\mathbb{Z}^d$ action (or $\mathcal{T}$ an $\mathbb{R}^d$ action).

(1) Let $\mathcal{E}^e_T$ (or $\mathcal{E}^e_{\mathcal{T}}$) be the set of $L \in \mathbb{G}_{e,d}$ so that T (or $\mathcal{T}$) is ergodic in the direction L.
(2) Let $\mathcal{W}^e_T$ (or $\mathcal{W}^e_{\mathcal{T}}$) be the set of $L \in \mathbb{G}_{e,d}$ so that T (or $\mathcal{T}$) is weak mixing in the direction L.

We define the set of *rational e-dimensional directions*, denoted by $\mathbb{G}^{\mathbb{Q}}_{e,d}$, to be those $L \in \mathbb{G}_{e,d}$ for which there is a choice of basis over $\mathbb{R}^d$ consisting of vectors in $\mathbb{Q}^d$. In this case there is a corresponding subgroup of $\mathbb{Z}^d$ and we show that Definition 1.2 is equivalent to the usual notions of T restricted to the subgroup L of $\mathbb{Z}^d$, T_L, being a weak mixing or ergodic action.

Theorem 1.4. *Let T be a $\mathbb{Z}^d$ action on (X,μ) and let $\tilde{\mathcal{T}}$ be its unit suspension. Let $L \in \mathbb{G}^{\mathbb{Q}}_{e,d}$ be a rational direction. Then $\tilde{\mathcal{T}}_L$ is ergodic (or weak mixing) if and only if T_L is ergodic (or weak mixing).*

1.2. **Characterizing directional ergodicity and weak mixing spectrally for $\mathbb{R}^d$ actions.** For any $\mathbb{R}^d$ action $\mathcal{T}$, it is clear that if $\mathcal{T}$ is not ergodic, then $\mathcal{E}^e_{\mathcal{T}} = \varnothing$, for all dimensions e. Similarly, if it is not weak mixing then $\mathcal{W}^e_{\mathcal{T}} = \varnothing$, for all $1 \le e \le d$. For suppose, if f is an eigenfunction for $\mathcal{T}$ with eigenvalue $\vec{\alpha}$. Then for all $\vec{v} \in \mathbb{R}^d$ the function f is an eigenfunction for the $\mathbb{R}$ action $\{T^{t\vec{v}}\}_{t\in\mathbb{R}}$ with eigenvalue $\vec{\alpha}\cdot\vec{v}$. Therefore, we see that if $\mathcal{T}$ is not weak mixing, then directions in $\text{Spec}(\mathcal{T})^{\perp}$ are not contained in $\mathcal{E}^e_{\mathcal{T}}$.

This is a special case of a more general phenomenon and related to the following result in [9].

Proposition 1.5. *Let $\mathcal{T}_0$ be an ergodic, measure preserving $\mathbb{R}^d$ action. and let σ_0 on $\mathbb{R}^d$ be a measure of maximal type for $\mathcal{T}$ without the atom at 0 and $L \in \mathbb{G}_{e,d}$. If $\mathcal{T}_L$ is not ergodic then $\sigma_0(L^{\perp}) > 0$.*

Now assume $\mathcal{T}$ is weak mixing as an $\mathbb{R}^d$ action. We show that in this case if $\mathcal{T}$ is not weak mixing in a direction $L \in \mathbb{G}_{e,d}$, then the spectral measure of $\mathcal{T}$ must give positive measure to an e dimensional plane in $\mathbb{R}^d$, and that in fact the flow must fail to be ergodic in that direction. The main result is the following.

Theorem 1.6. *Let $\mathcal{T}$ be a measure preserving, weak mixing $\mathbb{R}^d$ action. There is $L \in \mathbb{G}_{e,d}$ so that $L \notin \mathcal{W}^e_T$, with $e < d$ if and only if there exists $\vec{\omega} \in \mathbb{R}^d$ so that $\tilde{\sigma}(L^{\perp} + \vec{\omega}) > 0$, where $\tilde{\sigma}$ denotes the maximal spectral type of $\mathcal{T}$. Furthermore $\mathcal{W}^e_{\mathcal{T}} = \mathcal{E}^e_{\mathcal{T}}$ for all $1 \le e \le d$.*

1.3. **The maximal spectral type of T and of its unit suspension.** The results outlined in the previous section indicate that the absence of directional ergodicity or weak mixing for a $\mathbb{Z}^d$ action T can be detected from the structure of the spectrum of its unit suspension $\tilde{\mathcal{T}}$, restricted to $\mathcal{H}$. Here we show that the two spectral types are related, and thus we can define directional ergodicity and weak mixing intrinsically using the maximal spectral type of T as well.

Recall that the maximal spectral type of T, denoted by σ, is a (nonzero) finite measure on $\mathbb{T}^d$. We denote by $\tilde{\sigma}$ any measure on $\mathbb{R}^n$ that is finite and equivalent to $\sigma * \delta_{\mathbb{Z}^d}$, obtained by summing the translations of σ (viewed as a measure on $\mathbb{T}^d$) to each square $\prod_{j=1}^{d}[n_j, n_j+1)$ for $\vec{n} = (n_1, n_2, \dots, n_d)^t \in \mathbb{Z}^d$. Such a measure is given by $\rho \cdot (\sigma * \delta_{\mathbb{Z}^d})$

where $\rho(\vec{s}) > 0$ is chosen so that $\int_{\mathbb{R}^d} \rho\, d(\sigma * \delta_{Z^d}) < \infty$. In particular, this insures that $\tilde{\sigma}$ is a finite measure on $\mathbb{R}^d$.

We prove the following structure theorem for the maximal spectral type of $\tilde{\mathcal{T}}$, the unit suspension of T.

Theorem 1.7. *Let T be a $\mathbb{Z}^d$ action and let $\tilde{\mathcal{T}}$ be its unit suspension. If σ is the maximal spectral type of T then $\tilde{\sigma}$ is the maximal spectral type of $\tilde{\mathcal{T}}$. Moreover, if σ_0 is σ without its atom at $\vec{0}$, then $\tilde{\sigma}_0$ is the maximal spectral type of $\mathcal{U}_{\tilde{\mathcal{T}}}|_{\mathcal{H}}$, where $\mathcal{U}_{\tilde{\mathcal{T}}}$ is the Koopman operator associated to $\tilde{\mathcal{T}}$.*

The proof is an application of the following result, which is of independent interest.

Theorem 1.8. *Let $f \in L^2(X,p)$ be a function of maximal spectral type for $\mathcal{T}$ and let σ_f be the corresponding measure of maximal spectral type. Let $g \in L^2(\mathbb{T}^d, \lambda^d)$ be such that for all $\vec{s}, \vec{n} \in \mathbb{Z}^d$,*

$$\hat{g}(\vec{n}) = \int_{\mathbb{T}^d} g(\vec{s}) e^{2\pi i \vec{s}\cdot\vec{n}} d\lambda^n \neq 0,$$

i.e. $g(\vec{s}) = \sum_{\vec{n}\in\mathbb{Z}^d} \hat{g}(\vec{n}) e^{2\pi i \boldsymbol{r}\cdot\vec{n}}$ in $L^2(\mathbb{T}^d, \lambda^n)$, where $\sum_{\vec{n}\in\mathbb{Z}^d} |\hat{g}(\vec{n})|^2 < \infty$ and $\hat{g}(\vec{n}) \neq 0$. Then the function $F(x,\vec{s}) = f(x)g(\vec{s})$ is a function of maximal spectral type for $\tilde{\mathcal{T}}$, and the corresponding measure of maximal spectral type satisfies

$$\sigma_F = \left(\sum_{\vec{n}\in\mathbb{Z}^d} \hat{g}(\vec{n}) (\mathrm{sinc}_d^2 \circ R_{-\vec{n}})(\cdot) \right) \tilde{\sigma}_f. \tag{1}$$

Combining Theorems 1.6 and 1.8 we have the following.

Theorem 1.9. *Let T be a measure preserving, weak mixing $\mathbb{Z}^d$ action. There is $L \in \mathbb{G}_{e,d}$ so that $L \notin \mathcal{W}_T^e$, with $e < d$ if and only if there exists $\vec{\omega} \in \mathbb{T}^d$ so that $\sigma(L^\perp + \vec{\omega}) > 0$, where σ denotes the maximal spectral type of T. Furthermore $\mathcal{W}_T^e = \mathcal{E}_T^e$ for all $1 \le e \le d$.*

1.4. **The structure of $\mathcal{E}_T^e$ and $\mathcal{W}_T^e$.**

Theorem 1.9 now allows us to use L^2 arguments to deduce some limitations on the structure of $\mathcal{E}_T^e$ and $\mathcal{W}_T^e$. Let $\mathbb{G}_d = \cup_{e=1}^d \mathbb{G}_{e,d}$. We make the following definition.

Definition 1.10. Let $L \in \mathbb{G}_{e,d}$. The set $\mathbb{G}(L) \subset \mathbb{G}_d$ is defined to be those directions that are contained within L. In other words,

$$\mathbb{G}(L) = \{L' \in \mathbb{G}_{i,d} : i \le e \text{ and } L' \subset L\}.$$

Two directions $L, L' \in \mathbb{G}_d$ are said to be *independent* if $L \bigcap L' = \vec{0}$.

In [9], Theorem 1, the authors use their spectral characterization of non-ergodic directions to show that an ergodic $\mathbb{R}^d$ action can have at most countably many independent directions that are non-ergodic. Combining their techniques with Theorem 1.6 and Theorem 1.7 we have the following result.

Theorem 1.11. *Let T be a measurable and measure preserving $\mathbb{Z}^d$ action on a Lebesgue probability space (X,μ). If T is not ergodic, then $\mathcal{E}_T = \varnothing$. If T is ergodic, then there is a set of at most countably many independent directions $\{L_i\}$ so that $\mathcal{E}_T = \mathbb{G}_d \smallsetminus \bigcup_{i=1}^{\infty} \mathbb{G}(L_i)$. If T is ergodic but not weak mixing, then $\mathcal{W}_T = \varnothing$. If T is weak mixing, then there is a set of at most countably many independent directions $\{L_i\}$ so that $\mathcal{W}_T = \mathcal{E}_T = \mathbb{G}_d \smallsetminus \bigcup_{i=1}^{\infty} \mathbb{G}(L_i)$.*

We note that these structure results for $\mathcal{E}_T$ and $\mathcal{W}_T$ are in strong contrast with directional recurrence and rigidity in the infinite measure preserving case. In particular, for T a σ-finite infinite measure preserving and ergodic $\mathbb{Z}^d$ action, the set of reccurent directions for T, denoted by $\mathcal{R}_T$, has to be a G_δ but there exist examples for which $\mathcal{R}_T = \varnothing$ [5]. Similarly the set of rigid directions of an infinite measure preserving $\mathbb{Z}^d$ action is a G_δ [3].

1.5. **Constructing examples: realizing different types of $\mathcal{W}_T$ and $\mathcal{E}_T$.** Let T denote the weak mixing $\mathbb{Z}^d$ action constructed by Bergelson and Ward with the property that $\mathcal{E}_T^1 \subset \mathbb{G}_d \smallsetminus \mathbb{Z}^d$. Theorem 1.11 shows that, from the point of view of cardinality, $\mathcal{E}_T^1$ cannot miss too many more directions. We show, in fact, that for this action T there are no other directions of non-ergodicity.

Proposition 1.12. *Let T denote the Bergelson-Ward weak mixing $\mathbb{Z}^d$ action. Then*

$$\mathcal{E}_T^{d-1} = \mathcal{W}_T^{d-1} = \mathbb{G}_{d-1,d} \smallsetminus \mathbb{G}_{d-1,d}^{\mathbb{Q}}.$$

We realize other types of sets $\mathcal{E}_T$ and $\mathcal{W}_T$ using the Gaussian measure space construction (GMC). In what follows we say T is a Gaussian $\mathbb{Z}^d$ action if it is given by a GMC.

Theorem 1.13. *Given any countable collection $\{L_i\}$ of directions in $\mathbb{G}_{d-1,d}$, there is a measurable, measure preserving, and weak mixing Gaussian $\mathbb{Z}^d$ action T with the property that*

$$\mathcal{W}_T^{d-1} = \mathcal{E}_T^{d-1} = \mathbb{G}_{d-1,d} \smallsetminus \{L_i\}.$$

Furthermore, for $e < d-1$, $L \notin \mathcal{W}_T^e = \mathcal{E}_T^e$ if and only if $L \subset L_i$ for some i.

Once we choose a direction of non-ergodicity, due to the nature of the measure of maximal spectral type of a GMC, we are constrained as to how much freedom remains. We introduce some terminology for ease of exposition.

Definition 1.14. Let $e < d$, $L \in \mathbb{G}_{e,d}$, and $\vec{a} \in \mathbb{R}^d$. A measure μ on $\mathbb{R}^d$ is an e-dimensional wall along $L + \vec{a}$ if it is the push forward to $L + \vec{a}$ of a Borel measure μ^e on $\mathbb{R}^e$ that has no walls of dimension $e' < e$.

An e dimensional wall on $\mathbb{R}^d$ is then a measure that is supported along a genuinely e-dimensional subspace of $\mathbb{R}^d$. In other words, lower dimensional subsets of $L + \vec{a}$ have zero measure under a wall on $L + \vec{a}$. We will use such measures to construct our Gaussian example. The next proposition shows that if $d - 1 > e_1 > e_2$, then the convolution of e_1 and e_2 dimensional walls will be a wall of dimension greater than e_1, introducing some additional directional non-ergodicity for the Gaussian action.

Proposition 1.15. *Let μ and ν be measures on $\mathbb{R}^d$ that are e_1 and e_2-dimensional walls along $L_1 + \vec{a}_1$ and $L_2 + \vec{a}_2$ respectively, with $e_1 \geq e_2$. Let*

$$k = e_2 - dim\left(L_1 \cap L_2\right).$$

Then $L_1 + L_2 \in \mathbb{G}_{e_1+k,d}$ and $\mu * \nu$ is an $e_1 + k$ dimensional wall on $L_1 + L_2 + \vec{a}_1 + \vec{a}_2$.

The next result shows that we can construct a Gaussian $\mathbb{Z}^d$ action that corresponds to any directional behavior that is possible under this constraint.

Theorem 1.16. *Given $L \in \mathbb{G}_{e,d}$ with $e < d$, there is a measurable, measure preserving, and weak mixing $\mathbb{Z}^d$ action T with the property that*

(1) $\mathcal{W}_T^e = \mathcal{E}_T^e = \mathbb{G}_{e,d} \setminus L$,
(2) $\mathcal{W}_T^{e'} = \mathcal{E}_T^{e'} = \mathbb{G}_{e',d}$ *for all* $d > e' > e$, *and*
(3) *for* $e' < e$ *and* $L \in \mathbb{G}_{e',d}$, *we have* $L' \notin \mathcal{W}_T^{e'} = \mathcal{E}_T^{e'}$ *if and only if* $L' \subset L$.

1.6. **Other ways to define directional properties: embeddings.** Given a $\mathbb{Z}^d$ action T we say that an $\mathbb{R}^d$ action $\mathcal{T}$, acting on the same space as T, is an *embedding* of T if $\mathcal{T}^{\vec{n}} = T^{\vec{n}}$ for all $\vec{n} \in \mathbb{Z}^d$. If T has an embedding $\mathcal{T}$, then we could also define the directional ergodicity and weak mixing of T in a direction L by the corresponding property of $\mathcal{T}$ restricted to the subgroup L. A natural question to ask is if there is any relationship between the directional behavior of an embedding $\mathcal{T}$

and the unit suspension $\tilde{\mathcal{T}}$. We first note that $\tilde{\mathcal{T}}$ is in fact a cover of all embeddings of T.

Proposition 1.17. *Let T be a $\mathbb{Z}^d$ action and $\tilde{\mathcal{T}}$ its unit suspension. If the $\mathbb{R}^d$ action $\mathcal{T}$ is an embedding of T, then $\mathcal{T}$ is a factor of $\tilde{\mathcal{T}}$.*

The following result now follows immediately.

Proposition 1.18. *Let T be a $\mathbb{Z}^d$ action and $\mathcal{T}$ an embedding of T. For all $L \in \mathbb{G}_{e,d}$, if T is directionally ergodic (directionally weak mixing) in the direction L then $\mathcal{T}_L$ is ergodic (weak mixing).*

In this section we show by example that ergodicity of the $\mathbb{Z}^d$ action is a necessary condition for the converse statement to hold for directional ergodicity, while the converse result for directional weak mixing holds with no mixing assumptions on the $\mathbb{Z}^d$ action itself.

Theorem 1.19. *Let T be a $\mathbb{Z}^d$ action and the $\mathbb{R}^d$ action $\mathcal{T}$ be an embedding of T. Then T is directionally weak mixing in a direction $L \in \mathbb{G}_{e,d}$ if and only if $\mathcal{T}_L$ is weak mixing. If, in addition T is ergodic as a $\mathbb{Z}^d$ action, then T is directionally ergodic in a direction L if and only if $\mathcal{T}_L$ is ergodic.*

1.7. **Genericity results.**

Proposition 1.20. *There is rigid, weakly mixing action of $\mathbb{Z}^2$, as measure-preserving transformations of a probability space, which is also weakly mixing in all directions.*

Proposition 1.21. *The generic measure preserving action of $\mathbb{Z}^d$ on a Lebesgue probability space is weak mixing, and weak mixing in all directions.*

We note that Proposition 1.21 follows from Ryzhikov's work in [12]. Here we provide an alternative approach to proving genericity results using the GMC construction and Fourier techniques.

References

[1] V. Bergelson and T. Ward. Private communication.

[2] Mike Boyle and Douglas Lind. Expansive subdynamics. *Trans. Amer. Math. Soc.*, 349(1):55–102, 1997.

[3] Alexandre I. Danilenko. Directional reccurence and directional rigidity for infinite measure preserving actions of nilpotent lattices. Preprint.

[4] S. Ferenczi and B. Kamiński. Zero entropy and directional Bernoullicity of a Gaussian $\mathbf{Z}^2$-action. *Proc. Amer. Math. Soc.*, 123(10):3079–3083, 1995.

[5] Aimee S. A. Johnson and Ayşe A. Şahin. Directional recurrence for infinite measure preserving $\mathbb{Z}^d$ actions. *Ergodic Theory and Dynamical Systems.* Published electronically, August 2014.

[6] John Milnor. On the entropy geometry of cellular automata. *Complex Systems*, 2(3):357–385, 1988.
[7] Kyewon K. Park. Continuity of directional entropy. *Osaka J. Math.*, 31(3):613–628, 1994.
[8] Kyewon Koh Park. On directional entropy functions. *Israel J. Math.*, 113:243–267, 1999.
[9] Charles Pugh and Michael Shub. Ergodic elements of ergodic actions. *Compositio Math.*, 23:115–122, 1971.
[10] E. Arthur Robinson, Jr. and Ayşe A. Şahin. Rank-one $\mathbb{Z}^d$ actions and directional entropy. *Ergodic Theory Dynam. Systems*, 31(1):285–299, 2011.
[11] Daniel J. Rudolph. The second centralizer of a Bernoulli shift is just its powers. *Israel J. Math.*, 29(2-3):167–178, 1978.
[12] V. V. Ryzhikov. Factors, rank, and embedding of a generic $\mathbb{Z}^n$-action in an $\mathbb{R}^n$-flow. *Uspekhi Mat. Nauk*, 61(4(370)):197–198, 2006.
[13] Ya. G. Sinaĭ. An answer to a question by J. Milnor. *Comment. Math. Helv.*, 60(2):173–178, 1985.
[14] T. Ward. Three results on mixing shapes. *New York J. Math.*, 3A(Proceedings of the New York Journal of Mathematics Conference, June 9–13, 1997):1–10 (electronic), 1997/98.

(Robinson) Department of Mathematics, George Washington University, Washington, DC 20052

(Rosenblatt) Department of Mathematics, University of Illinois at Urbana-Champaign, Urbana, IL 61801

(Şahin) Department of Mathematical Sciences, DePaul University, 2320 N. Kenmore Ave., Chicago, IL 60626