

LIPSCHITZ SPACES AND CALDERÓN-ZYGMUND OPERATORS ASSOCIATED TO NON-DOUBLING MEASURES

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ABSTRACT. In the setting of a metric measure space (\mathbb{X}, d, μ) with an n -dimensional Radon measure μ , we show that Lipschitz spaces are preserved by Calderón-Zygmund operators T associated to the measure μ . Also, for the Euclidean space \mathbb{R}^d with an arbitrary Radon measure μ , we give several characterizations of Lipschitz spaces on the support of μ , $\mathcal{L}ip(\alpha, \mu)$, in terms of mean oscillations involving μ . This allows us to view the “regular” BMO space of X . Tolsa as a limit case for $\alpha \rightarrow 0$ of the spaces $\mathcal{L}ip(\alpha, \mu)$.

1. INTRODUCTION

The present paper is devoted, on the one hand, to study the invariance of Lipschitz spaces under Calderón-Zygmund operators associated to an n -dimensional Radon measure μ .

We do that in section 2, in the context of a **metric measure space** (\mathbb{X}, d, μ) with an n -dimensional measure, that is a measure satisfying condition (2.1). This allows, in particular, non-doubling measures.

A second aim is to show that, for any Radon measure in the Euclidean space \mathbb{R}^d , the Lipschitz spaces can be characterized by a host of integral oscillation conditions similar to the regular BMO condition introduced by Tolsa. This shows that the regular BMO space of Tolsa is a limit case of the natural Lipschitz spaces associated to the measure.

The study of Calderón-Zygmund operators associated to an n -dimensional Radon measure was carried out, in the Lebesgue spaces, by Nazarov, Treil and Volberg (see [NTV1, NTV2]) and also by Tolsa

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(see [To1, To2]). Further results, dealing with BMO and H^1 and providing boundedness criteria in the spirit of the $T(1)$ or $T(b)$ theorems, were obtained as well (see [NTV3, MMNO, To3])

In [GG] we have also studied, on metric spaces, the theory of fractional integral operators associated to an n -dimensional Radon measure μ on Lebesgue spaces and Lipschitz spaces.

In a previous version of this paper we were only considering n -dimensional Radon measures, which are the ones we are mainly interested on and the only ones for which we can prove the boundedness of Calderón-Zygmund operators. However, Xavier Tolsa made the observation, that we gratefully acknowledge, that Theorem 3.3 and its proof were valid for a general Radon measure, since the n -dimensional nature of the measure, was never used.

We also want to thank Prof. Peter Constantin for a valuable observation about Theorem 2.5.

2. CALDERÓN-ZYGMUND OPERATORS

In this section, (\mathbb{X}, d, μ) will be a **metric measure space** (that is, d is a distance on \mathbb{X} and μ is a Borel measure on \mathbb{X}), such that, for every ball

$$B(x, r) = \{y \in \mathbb{X} : d(x, y) < r\}, \quad x \in \mathbb{X}, \quad r > 0,$$

we have

$$(2.1) \quad \mu(B(x, r)) \leq Cr^n,$$

where n is some fixed positive real number and C is independent of x and r .

We shall also refer to condition (2.1) by saying that the measure μ is n -dimensional.

Whenever we refer to “the ball B ”, we shall understand that we have chosen for it a fixed center and a fixed radius. That way, it makes sense to say that if B is a ball and k is a positive real number, we shall denote by kB the ball having the same center as B and radius k times that of B .

From now on, we shall assume that $\mu(\mathbb{X}) = \infty$.

We shall use below two basic lemmas from section 2 of [GG], which allow us to bound the integrals against an n -dimensional measure of potential kernels on balls or complements of balls. For completeness, we state those two lemmas together here without the proofs.

Lemma 2.1. *For every $\gamma > 0$ and every $r > 0$:*

$$\int_{B(x,r)} \frac{1}{d(x,y)^{n-\gamma}} d\mu(y) \leq Cr^\gamma \text{ and } \int_{\mathbb{X} \setminus B(x,r)} \frac{1}{d(x,y)^{n+\gamma}} d\mu(y) \leq Cr^{-\gamma}$$

Definition 2.2. Given $\beta \in]0, 1[$, we shall say that the function $f : \mathbb{X} \rightarrow \mathbb{C}$ satisfies a Lipschitz condition of order β provided

$$(2.2) \quad |f(x) - f(y)| \leq Cd(x,y)^\beta \text{ for every } x, y \in \mathbb{X}$$

and the smallest constant in inequality (2.2) will be denoted by $\|f\|_{\mathcal{L}ip(\beta)}$. It is easy to see that the linear space of all Lipschitz functions of order β , modulo constants, becomes, with the norm $\|\cdot\|_{\mathcal{L}ip(\beta)}$, a Banach space, which we shall call $\mathcal{L}ip(\beta)$.

Next, we define the class of singular kernels that we consider in this paper.

Definition 2.3. A singular kernel on a metric measure space (\mathbb{X}, d, μ) with μ n -dimensional, will be a measurable function $K(x, y)$ on $\mathbb{X} \times \mathbb{X} \setminus \{x = y\}$ satisfying the following conditions:

- (1) $|K(x, y)| \leq \frac{A_1}{d(x,y)^n}$,
- (2) $|K(x_1, y) - K(x_2, y)| \leq \frac{A_2 d(x_1, x_2)^\delta}{d(x_1, y)^{n+\delta}}$ for $2d(x_1, x_2) \leq d(x_1, y)$,
- (3) $\left| \int_{\varepsilon < d(x,y) < R} K(x, y) d\mu(y) \right| \leq B$ for all $0 < \varepsilon < R$.
- (4) $\lim_{\varepsilon \rightarrow 0} \int_{\varepsilon < d(x,y) < R} K(x, y) d\mu(y)$ exists for μ -almost every point x .

We define the truncated kernels

$$K_\varepsilon(x, y) = K(x, y) \chi_{\{d(x,y) > \varepsilon\}}(x, y).$$

Definition 2.4. For $f \in \mathcal{L}ip(\alpha)$, $0 < \alpha < \delta \leq 1$, we define

$$\tilde{T}_\varepsilon f(x) = \int_{\mathbb{X}} (K_\varepsilon(x, y) - K_1(x_0, y)) f(y) d\mu(y)$$

and then

$$\tilde{T}f(x) = \lim_{\varepsilon \rightarrow 0} \tilde{T}_\varepsilon f(x)$$

Note that it follows from the properties of $K(x, y)$ and Lemma 2.1 that the limit exists for μ -almost everywhere. Indeed

$$\begin{aligned} \tilde{T}_\varepsilon f(x) &= \int_{\varepsilon < d(x,y) < R_x} K_\varepsilon(x, y) f(y) d\mu(y) \\ &\quad + \int_{\mathbb{X}} (K_{R_x}(x, y) - K_1(x_0, y)) f(y) d\mu(y) \end{aligned}$$

for appropriate R_x . The second integral converges and the first one can be written as

$$\int_{\varepsilon < d(x,y) < R_x} K_\varepsilon(x,y) (f(y) - f(x)) \, d\mu(y) + \left(\int_{\varepsilon < d(x,y) < R_x} K_\varepsilon(x,y) \, d\mu(y) \right) f(x),$$

which converges μ -almost everywhere.

Theorem 2.5. *Let K be a singular kernel as above and let \tilde{T} be the corresponding singular integral operator. Let $0 < \alpha < \delta \leq 1$. Then \tilde{T} is a bounded operator on $\mathcal{L}ip(\alpha)$, if and only if $\tilde{T}(1)$ is constant.*

Proof. The necessity of $\tilde{T}(1) = \text{constant}$ follows from the fact that $\|1\|_{\mathcal{L}ip(\alpha)} = 0$. We shall show next, that this condition is also sufficient.

$$\begin{aligned} \tilde{T}f(x_1) - \tilde{T}f(x_2) &= \lim_{\varepsilon \rightarrow 0} \left\{ \tilde{T}_\varepsilon f(x_1) - \tilde{T}_\varepsilon f(x_2) \right\} \\ &= \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{X}} (K_\varepsilon(x_1, y) - K_\varepsilon(x_2, y)) f(y) \, d\mu(y) \end{aligned}$$

The same computation with $f = 1$ shows that

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{X}} (K_\varepsilon(x_1, y) - K_\varepsilon(x_2, y)) \, d\mu(y) = 0$$

Thus, we can write

$$\begin{aligned} \tilde{T}f(x_1) - \tilde{T}f(x_2) &= \\ &= \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{X}} (K_\varepsilon(x_1, y) - K_\varepsilon(x_2, y)) (f(y) - f(x_1)) \, d\mu(y) \end{aligned}$$

Now let $r = d(x_1, x_2)$ and take $\varepsilon < r$. Split the integral in the limit above as $I_1 + I_2$, where I_1 is the integral over $B(x_1, 3r)$ and I_2 the integral over its complement.

Note that in I_2 we can write K in place of K_ε and also, since we have $d(x_1, y) > 2d(x_1, x_2)$, we can apply condition (2) to get, as an estimate,

$$\begin{aligned} |I_2| &\leq \int_{d(x_1, y) > 3r} \frac{d(x_1, x_2)^\delta}{d(x_1, y)^{n+\delta}} d(y, x_1)^\alpha \, d\mu(y) \\ &\leq C d(x_1, x_2)^\delta r^{\alpha-\delta} \|f\|_{\mathcal{L}ip(\alpha)} \leq C \|f\|_{\mathcal{L}ip(\alpha)} d(x_1, x_2)^\alpha \end{aligned}$$

Next, write $I_1 = J_1 - J_2 + J_3$, where

$$J_1 = \int_{B(x_1, 3r)} K_\varepsilon(x_1, y) (f(y) - f(x_1)) \, d\mu(y)$$

$$J_2 = \int_{B(x_1, 3r)} K_\varepsilon(x_2, y) (f(y) - f(x_2)) \, d\mu(y) \text{ and}$$

$$J_3 = \int_{B(x_1, 3r)} K_\varepsilon(x_2, y) (f(x_2) - f(x_1)) \, d\mu(y).$$

Both $|J_1|$ and $|J_2|$ can be estimated using the fact that $f \in \mathcal{L}ip(\alpha)$ and Lemma 2.1 and we find that they are bounded by $Cr^\alpha \|f\|_{\mathcal{L}ip(\alpha)}$.

To estimate the third one, we rewrite it as

$$J_3 = \left(\int_{\{\varepsilon < d(x_2, y) < r\}} K_\varepsilon(x_2, y) \, d\mu(y) + \int_{\{r \leq d(x_2, y); d(x_1, y) < 3r\}} K_\varepsilon(x_2, y) \, d\mu(y) \right) (f(x_2) - f(x_1))$$

The first integral is bounded in absolute value by B . And the second one is bounded by

$$\int_{\{r \leq d(x_2, y) < 4r\}} |K_\varepsilon(x_2, y)| \, d\mu(y) \leq CA_1 4^n.$$

We get $|J_3| \leq Cr^\alpha \|f\|_{\mathcal{L}ip(\alpha)}$.

Finally, collecting the estimates, we have

$$\left| \tilde{T}f(x_1) - \tilde{T}f(x_2) \right| \leq |I_1| + |I_2| \leq C \|f\|_{\mathcal{L}ip(\alpha)} d(x_1, x_2)^\alpha$$

□

3. CHARACTERIZATION OF LIPSCHITZ SPACES

All throughout this section, μ will be a fixed Radon measure on \mathbb{R}^d . This is all that we need for the results we prove in this section, in particular for Theorem 3.3. Only for Remark 3.8 we will need to assume that the measure is n -dimensional.

From now on, all balls that we consider will be centered at points in the support of μ .

In order to prove the main theorem of this section we will need the following known definition and lemma (see [To3])

Definition 3.1. Let β be a fixed constant. A ball B is called β -doubling if

$$\mu(2B) \leq \beta\mu(B).$$

Lemma 3.2. *Let $f \in L^1_{\text{loc}}(\mu)$. If $\beta > 2^d$, then, for almost every x with respect to μ , there exists a sequence of β -doubling balls $B_j = B(x, r_j)$ with $r_j \rightarrow 0$, such that*

$$\lim_{j \rightarrow \infty} \frac{1}{\mu(B_j)} \int_{B_j} f(y) \, d\mu(y) = f(x).$$

Proof. We will show that for almost every x with respect to μ there is a β -doubling ball centered at x with radius as small as we wish. This fact, combined with the differentiation theorem, completes the proof of the lemma.

We know that for almost every x with respect to μ

$$(3.1) \quad \lim_{r \rightarrow 0} \frac{\mu(B(x, r))}{r^d} > 0$$

(differentiation of μ with respect to Lebesgue measure, see [M]). Now for x satisfying (3.1) take $B = B(x, r)$ and assume that none of the balls $2^{-k}B$, $k \geq 1$, is β -doubling. Then it is easy to see that $\mu(B) > \beta^k \mu(2^{-k}B)$ for all $k \geq 1$. Therefore

$$\frac{\mu(2^{-k}B)}{(2^{-k}r)^d} < \left(\frac{2^d}{\beta}\right)^k \frac{\mu(B)}{r^d}.$$

Note that, since $\beta > 2^d$, the right hand side tends to zero for $k \rightarrow \infty$, which is a contradiction. □

Now we can state and prove the main result of this section.

Theorem 3.3. *For a function $f \in L^1_{\text{loc}}(\mu)$, the conditions I, II, and III below, are equivalent*

- (I) *There exist some constant C_1 and a collection of numbers f_B , one for each ball B , such that these two properties hold: For any ball B with radius r*

$$(3.2) \quad \frac{1}{\mu(2B)} \int_B |f(x) - f_B| \, d\mu(x) \leq C_1 r^\alpha,$$

and for any ball U such that $B \subset U$ and $\text{radius}(U) \leq 2r$.

$$(3.3) \quad |f_B - f_U| \leq C_1 r^\alpha,$$

- (II) *There is a constant C_2 such that*

$$(3.4) \quad |f(x) - f(y)| \leq C_2 |x - y|^\alpha$$

for μ -almost every x and y in the support of μ .

(III) For any given p , $1 \leq p \leq \infty$, there is a constant $C(p)$, such that for every ball B of radius r , we have

$$(3.5) \quad \left(\frac{1}{\mu(B)} \int_B |f(x) - m_B(f)|^p d\mu(x) \right)^{1/p} \leq C(p)r^\alpha,$$

where $m_B(f) = \frac{1}{\mu(B)} \int_B f(y) d\mu(y)$ and also for any ball U such that $B \subset U$ and $\text{radius}(U) \leq 2r$.

$$(3.6) \quad |m_B(f) - m_U(f)| \leq C(p)r^\alpha,$$

In addition, the quantities: $\inf C_1$, $\inf C_2$, and $\inf C(p)$ with a fixed p are equivalent.

Proof. (I) \Rightarrow (II). Consider x as in the Lemma and let $B_j = B(x, r_j)$, $j \geq 1$, a sequence of β -doubling balls with $r_j \rightarrow 0$. We will show first that (3.2) implies

$$\lim_{j \rightarrow \infty} f_{B_j} = f(x).$$

It suffices to observe that

$$\begin{aligned} |m_{B_j}(f) - f_{B_j}| &\leq \frac{1}{\mu(B_j)} \int_{B_j} |f(y) - f_{B_j}| d\mu(y) \\ &\leq \frac{\mu(2B_j)}{\mu(B_j)} \frac{1}{\mu(2B_j)} \int_{B_j} |f(y) - f_{B_j}| d\mu(y) \leq \beta C_1 r_j^\alpha. \end{aligned}$$

Next, let x and y be two points as in the lemma. Take $B = B(x, r)$ any ball with $r \leq |x - y|$ and let $U = B(x, 2|x - y|)$. Now define $B_k = B(x, 2^k r)$, for $0 \leq k \leq \bar{k}$, where \bar{k} is the first integer such that $2^{\bar{k}} r \geq |x - y|$. Then

$$\begin{aligned} |f_B - f_U| &\leq \sum_{k=0}^{\bar{k}-1} |f_{B_k} - f_{B_{k+1}}| + |f_{B_{\bar{k}}} - f_U| \\ &\leq C_1 \sum_{k=0}^{\bar{k}} (2^k r)^\alpha \leq C' C_1 |x - y|^\alpha, \end{aligned}$$

where C' is independent of x and B .

A similar argument can be made for the point y with any ball $B' = B(y, s)$ such that $s \leq |x - y|$ and $V = B(y, 3|x - y|)$. Therefore

$$|f_B - f_{B'}| \leq |f_B - f_U| + |f_U - f_V| + |f_V - f_{B'}| \leq C'' C_1 |x - y|^\alpha.$$

Finally, take two sequences of β -doubling balls $B_j = B(x, r_j)$ and $B'_j = B(y, s_j)$ with $r_j \rightarrow 0$ and $s_j \rightarrow 0$. We have

$$|f(x) - f(y)| = \lim_{j \rightarrow \infty} |f_{B_j} - f_{B'_j}| \leq C'' C_1 |x - y|^\alpha.$$

(II) \Rightarrow (III). It is immediate. Note also that (II) \Rightarrow (I) is immediate as well.

(III) \Rightarrow (I). Define first $f_B = m_B(f)$. Then (3.3) is exactly (3.6).

In addition, the left hand side of (3.2) is less than or equal to the left hand side of (3.5).

This concludes the proof of the theorem. □

Remark 3.4. Theorem 3.3 is also true if the number 2 in condition (I) is replaced by any fixed $\rho > 1$. In that case, the proof uses (ρ, β) -doubling balls, that is, balls satisfying $\mu(\rho B) \leq \beta \mu(B)$. However this extension is not needed in our paper.

The idea of combining the mean oscillation condition with an extra condition as in (I) originates in the work of Tolsa [To3] on regular BMO , whereas the introduction of the ρ factor in (I) comes from [NTV3].

Definition 3.5. We shall call Lipschitz function of order α with respect to μ to a function, or rather the corresponding Lebesgue class in $L^1_{\text{loc}}(\mu)$, which satisfies any, and hence all, of the conditions of theorem 3.3.

The linear space of all Lipschitz functions of order α , with respect to μ , modulo constants, becomes, with the norm $\inf C_2$ of Theorem 3.3, a Banach space, which we shall call $\mathcal{L}ip(\alpha, \mu)$.

Remark 3.6. It is easy to see that $\mathcal{L}ip(\alpha, \mu)$ coincides with the space of Lipschitz functions of order α on the support of μ .

Note that by the extension theorem of Banach (see [B] or [Mi]¹), any Lipschitz function of order α with respect to μ has an extension to \mathbb{R}^d that is a Lipschitz function of order α with an equivalent norm.

Remark 3.7. For $0 < \alpha \leq 1$, a telescoping argument like the one used in the proof of (I) \Rightarrow (II) in Theorem 3.3 shows that (3.3) is equivalent to

$$(3.7) \quad |f_B - f_U| \leq C'_1 \text{radius}(U)^\alpha$$

for any two balls $B \subset U$.

¹In this reference, there is a note added in proof, to the effect that “Banach’s theorem mentioned above is probably due to McShane”

Remark 3.8. For this remark we further assume that μ is n -dimensional. Furthermore (3.7) is also equivalent to

$$(3.8) \quad |f_B - f_U| \leq C_1'' K_{B,U} \text{radius}(U)^\alpha,$$

for any two balls $B \subset U$, where $K_{B,U}$ is the constant introduced by X. Tolsa in [To3], given by

$$K_{B,U} = 1 + \sum_{j=1}^{N_{B,U}} \frac{\mu(2^j B)}{(2^j r)^n},$$

with $N_{B,U}$ equal to the first integer k such $2^k \text{radius}(B) \geq \text{radius}(U)$. Indeed (3.8) for comparable balls, that is, for $\text{radius}(U) \leq 2 \text{radius}(B)$, reduces to (3.3) because, in this case, $K_{B,U}$ is controlled by an absolute constant.

Note that (3.2) and (3.8) make sense also for $\alpha = 0$ and the space defined by them is the space $RBMO(\mu)$ of X. Tolsa (see [To3]). Therefore, the spaces $\mathcal{L}ip(\alpha, \mu)$, $0 < \alpha \leq 1$ can be seen as members of a family containing also $RBMO(\mu)$.

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