

[Ch 9] Counting Methods

1 Introduction

- In many discrete problems, we are confronted with the problem of counting. Here we develop tools which help us counting.
- Examples:
 - [9.1.2 (p. 519)] A pair of dice (with six sides) are thrown.
 - How many possible outcomes are there?
 - How many of them sum up to 6? What is the probability for that case?
 - [Theorem 9.1.1. The number of Elements in a List]

Theorem 9.1.1 The Number of Elements in a List

If m and n are integers and $m \leq n$, then there are $n - m + 1$ integers from m to n inclusive.

2 Basic Principles

2.1 Multiplication Rule

Theorem 9.2.1 The Multiplication Rule

If an operation consists of k steps and

the first step can be performed in n_1 ways,

the second step can be performed in n_2 ways *[regardless of how the first step was performed]*,

\vdots

the k th step can be performed in n_k ways *[regardless of how the preceding steps were performed]*,

then the entire operation can be performed in $n_1 n_2 \cdots n_k$ ways.

- Example 1:

A restaurant menu has 2 items for Appetizers, 3 items for main courses, and 4 items for beverages. If we list all possible dinners consisting of one appetizer, one main course and one beverage, how many different dinners can you make?

- Example 2:
 - a. How many strings of length 4 can be formed using the letters ABCDE if repetitions are allowed?
 - b. How about if repetitions are NOT allowed?
 - c. How many strings of part (a) begin with the letter B?

d. How many strings of part (a) do not begin with the letter B?

- Example 3: [9.2.6, p. 529] Counting the number of iterations of a nested loop.

How many times will the inner loop be iterated?

```
for i := 1 to 4
  for j := 1 to 3
    [Statements in body of inner loop.
     None contain branching statements
     that lead out of the inner loop.]
  next j
next i
```

2.2 Addition Rule

Theorem 9.3.1 The Addition Rule

Suppose a finite set A equals the union of k distinct mutually disjoint subsets A_1, A_2, \dots, A_k . Then

$$N(A) = N(A_1) + N(A_2) + \dots + N(A_k).$$

- Example 1: [9.3.1, p. 540]

A password consists of from one to three letters chosen from the 26 alphabets with repetitions allowed. How many different passwords are possible?

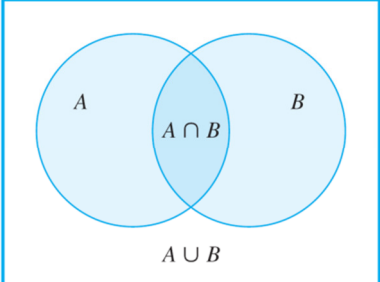


- Example 2:

In how many ways can we select two books from different subjects among 5 distinct computer science books, 3 distinct mathematics books and 2 distinct art books?

2.3 Inclusion/Exclusion Rule

- How to determine the number of elements in a union of sets when some of the sets **overlap**.

	<p>Theorem 9.3.3 The Inclusion/Exclusion Rule for Two or Three Sets</p> <p>If A, B, and C are any finite sets, then</p> $N(A \cup B) = N(A) + N(B) - N(A \cap B)$ <p>and</p> $N(A \cup B \cup C) = N(A) + N(B) + N(C) - N(A \cap B) - N(A \cap C) - N(B \cap C) + N(A \cap B \cap C).$
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- Example: [9.3.6, p. 546]

How many integers from 1 through 1,000 are multiples of 3 or multiples of 5?

More Examples

*** Multiplication Rule**

1. How many different bit strings of length seven are there?
2. How many different bit strings of length n are there?
3. How many different license plates are available if each plate contains a sequence of three letters followed by three digits?
4. How many functions are there from a set with m elements to one with n elements?
5. How many one-to-one functions are there from a set A with m elements to one B with n elements?
6. What is value of k after the following code has been executed?

```
k := 0
  for i1 := 1 to n1
    for i2 := 1 to n2
      :
      :
      for im := 1 to nm
        k := k + 1
```

*** Addition Rule**

1. Suppose that either a member of the CS faculty or a student who is CS major is chosen as a representative to a university committee. How many different choices are there for this representative if there are 37 members of the CS faculty and 83 CS majors and no one is both a faculty member and a student?
2. A student can choose a computer project from one of three lists. The three lists contain 23, 15, and 19 possible projects, respectively. No project is on more than one list. How many possible projects are there to choose from?
3. What is value of k after the following code has been executed?

```
k := 0
  for i1 := 1 to n1
    k := k + 1

  for i2 := 1 to n2
    k := k + 1
    :
  for im := 1 to nm
    k := k + 1
```

4. Each user on a computer system has a password, which is six to eight characters long, where each character is an uppercase letter or a digit. Each password must contain at least one digit. How many possible passwords are there?

*** Inclusion/Exclusion Rule**

1. How many bit strings of length eight either start with a 1 bit or end with two bits 00?

2. A computer company receives 350 applications from computer graduates for a job planning a line of new Web servers. Suppose that 220 of these people majored in CS, 147 msjored in business, and 51 majored both in CS and in business. How many of these applicants majored neither in CS nor in business?

3 Permutations and Combinations

3.1 Permutations

- A permutation of a set of objects is an ordering of the objects in a row. For example, the set of elements a, b, and c has six permutations.

abc acb cba bac bca cab

- Definition:** A **permutation** of n distinct elements x_1, \dots, x_n is an ordering of n elements x_1, \dots, x_n .
- In general, given a set of n objects, how many permutations does the set have?
 $\implies n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1 = n!$

Theorem 9.2.2

For any integer n with $n \geq 1$, the number of permutations of a set with n elements is $n!$.

- Example 1:** [9.2.8, p. 532] Permutations of the Letters in a Word
 - How many ways can the letters in the word COMPUTER be arranged in a row?

ANSWER: $8! = 40,320$

- How many ways can the letters in the word COMPUTER be arranged if the letters CO must remain next to each other (in order) as a unit?

ANSWER: CO M P U T E R, $7! = 5,040$

- If letters of the word COMPUTER are randomly arranged in a row, what is the probability that the letters CO remain next to each other (in order) as a unit?

ANSWER: $\frac{5040}{40320} = \frac{1}{8} = 0.125$, so 12.5%.

- Example 2:** [9.2.9, p. 532] Permutations of Objects Around a Circle

How many ways can six diplomats (A,B,C,D,E,F) be seated around circular table?

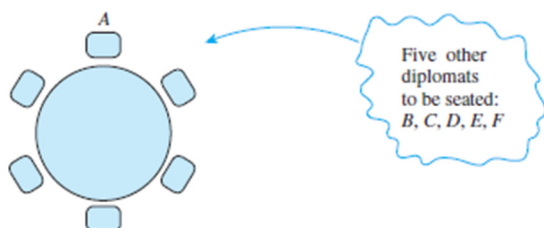


Figure 9.2.5

ANSWER: Since only relative position matters, you can start with any diplomat (say A), place that diplomat anywhere, and then place all others. So there are $5! = 120$ ways (i.e., $(n-1)!$ ways).

* r-Permutation

- Sometimes we want to consider an ordering of r elements selected from n available elements. Such an ordering is called an r -permutation.

• Definition

An **r -permutation** of a set of n elements is an ordered selection of r elements taken from the set of n elements. The number of r -permutations of a set of n elements is denoted $P(n, r)$.

Theorem 9.2.3

If n and r are integers and $1 \leq r \leq n$, then the number of r -permutations of a set of n elements is given by the formula

$$P(n, r) = n(n-1)(n-2) \cdots (n-r+1) \quad \text{first version}$$

or, equivalently,

$$P(n, r) = \frac{n!}{(n-r)!} \quad \text{second version.}$$

- Example 1: Given the set $\{a, b, c\}$, there are six ways to select two letters from the set and write them in order.

ab ac ba bc ca cb

$$\text{So } P(3, 2) = \frac{3!}{1!} = 3 \cdot 2 = 6$$

- Example 2: [9.2.11, p. 535]

How many different ways can three of the letters of the word BYTES be chosen and written in a row?

- Example 3: In how many ways can 7 distinct Martians and 5 distinct Jovians wait in line if no two Jovians stand together?

Hint: _ M₁ _ M₂ _ M₃ _ M₄ _ M₅ _ M₆ _ M₇ _

3.2 Combinations

- A selection of objects without regard to order is called a **combination**.
Example: “Given a set S with n elements, how many subsets of size r can be chose from S ?”
- Definition:

• **Definition**

Let n and r be nonnegative integers with $r \leq n$. An **r -combination** of a set of n elements is a subset of r of the n elements. As indicated in Section 5.1, the symbol

$$\binom{n}{r},$$

which is read “ n choose r ,” denotes the number of subsets of size r (r -combinations) that can be chosen from a set of n elements.

- And the number of combinations is defined as follows:

Theorem 9.5.1

The number of subsets of size r (or r -combinations) that can be chosen from a set of n elements, $\binom{n}{r}$, is given by the formula

$$\binom{n}{r} = \frac{P(n, r)}{r!} \quad \text{first version}$$

or, equivalently,

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} \quad \text{second version}$$

where n and r are nonnegative integers with $r \leq n$.

Note that $\binom{n}{r} = \binom{n}{n-r}$.

- Example 1: [9.5.4, p. 569]

Consider again the problem of choosing five members from a group of twelve to work as a team on a special project. How many distinct five-person teams can be chosen?

$$\text{ANSWER: } \binom{12}{5} = \frac{12!}{5!(12-5)!} = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7!}{(5 \cdot 4 \cdot 3 \cdot 2 \cdot 1) \cdot 7!} = 11 \cdot 9 \cdot 8 = 792$$

- Example 2:

In how many ways can we select a committee of 2 women and 3 men from a group of 5 distinct women and 6 distinct men?

- Example 3: [9.5.9, p. 575]

How many eight-bit strings have exactly three 1's?

3.3 Generalized Permutations and Combinations (with Repetitions)

* Permutation with repetition

- The number of r -permutations of a set of n objects with repetitions allowed is n^r .
- Example: The number of outcomes when a pair of (6-sided) dice is tossed is $6^2 = 36$.

* Combination with repetition

- When elements can be selected with repetition, the combinations are slightly different.

• Definition

An **r -combination with repetition allowed**, or **multiset of size r** , chosen from a set X of n elements is an unordered selection of elements taken from X with repetition allowed. If $X = \{x_1, x_2, \dots, x_n\}$, we write an r -combination with repetition allowed, or multiset of size r , as $[x_{i_1}, x_{i_2}, \dots, x_{i_r}]$ where each x_{i_j} is in X and some of the x_{i_j} may equal each other.

Theorem 9.6.1

The number of r -combinations with repetition allowed (multisets of size r) that can be selected from a set of n elements is

$$\binom{r+n-1}{r}.$$

This equals the number of ways r objects can be selected from n categories of objects with repetition allowed.

- Another, different definition:

Suppose that a sequence S of n items has n_1 identical objects of type 1, n_2 identical objects of type 2, ..., and n_k identical objects of type k . Then the number of ordering of S is

$$\frac{n!}{n_1! \cdot n_2! \cdot \dots \cdot n_k!}$$

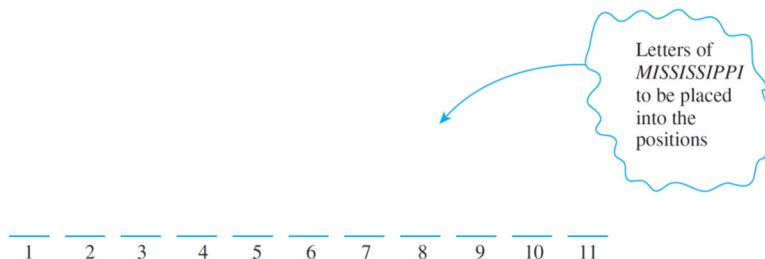
- Example1: [9.5.10, p. 576]

Consider various ways of ordering the letters in the word MISSISSIPPI:

IIMSSPISSIP, ISSSPMIIPIS, PIMISSSSIIP, and so on.

How many distinguishable orderings are there?

ANSWER:



Because copies of the same letter cannot be distinguished from one another, once the positions for a certain letter are known, then all copies of the letter can go into the positions in any order.

$$\left[\begin{array}{c} \text{number of ways to} \\ \text{position all the letters} \end{array} \right] = \binom{11}{4} \binom{7}{4} \binom{3}{2} \binom{1}{1}$$

Then by using the formula $\frac{n!}{n_1! \cdot n_2! \cdot \dots \cdot n_k!}$, we have $\frac{11!}{2! \cdot 4! \cdot 4! \cdot 1!} =$

34650.

- Example2: [9.6.2, p. 586]

A person giving a party wants to set out 15 assorted cans of soft drinks for his guests. He shops at a store that sells five different types of soft drinks.

- How many different selections of cans of 15 soft drinks can he make?
- If root beer is one of the types of soft drink, how many different selections include at least six cans of root beer?
- If the store has only five cans of root beer but at least 15 cans of each other type of soft drink, how many different selections are there?

- Example3: [9.6.4, p. 588] Counting Iterations of a Loop

How many times will the innermost loop be iterated when the algorithm segment below is implemented and run? (Assume n is a positive integer.)

```

for k := 1 to n
  for j := 1 to k
    for i := 1 to j
      [Statements in the body of the inner loop,
       none containing branching statements that lead
       outside the loop]
    next i
  next j
next k

```

ANSWER:

Construct a trace table for the values of k, j, and i for which the statements in the body of the innermost loop are executed. (See the table that follows.) Because i goes from 1 to j, it is always the case that $i \leq j$.

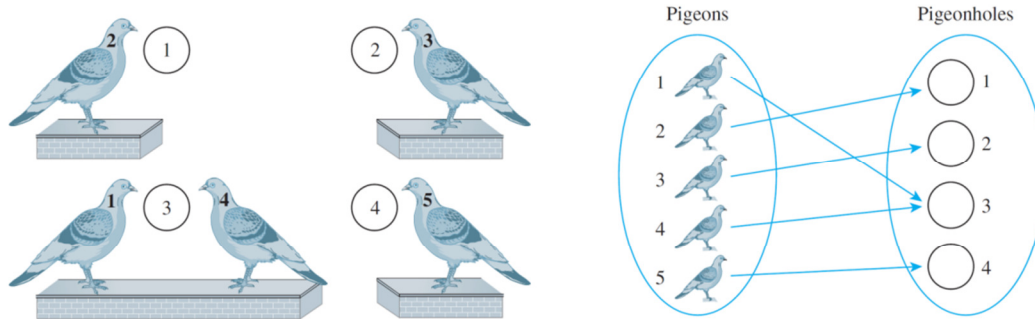
k	1	2			3					...	n					
j	1	1	2		1	2		3		...	1	2		...	n	
i	1	1	1	2	1	1	2	1	2	3	...	1	1	2	...	1

Similarly, because j goes from 1 to k, it is always the case that $j \leq k$.

[the rest to be worked out in the class]

4 Pigeonhole Principle

- The pigeonhole principle states that if n pigeons fly into m pigeonholes and $n > m$, then at least one hole must contain two or more pigeons.



- Application of the principle to mathematics:

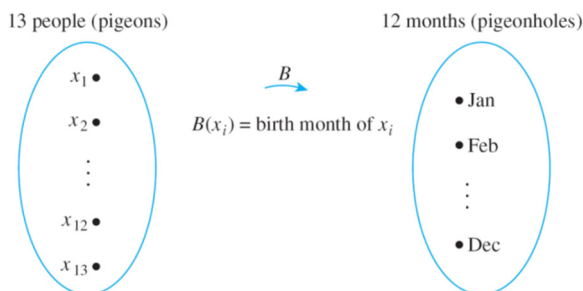
Pigeonhole Principle

A function from one finite set to a smaller finite set cannot be one-to-one: There must be at least two elements in the domain that have the same image in the co-domain.

Or another definition:

If f is a function from a finite set X to a finite set Y and $|X| > |Y|$, then $f(x_1) = f(x_2)$ for some $x_1, x_2 \in X$, $x_1 \neq x_2$.

- Example 1: [9.4.1. a, p. 554]
 - In a group of six people, must there be at least two who were born in the same month?
 - In a group of thirteen people, must there be at least two who were born in the same month? Why?



- Example 2:

An inventory consists of a list of 80 items, each marked with 'available' or 'unavailable'. There are 45 available items. Show that there are at least two available items in the list exactly nine items apart. (For example, available items at positions 13 and 22 or positions 69 and 78 satisfy the condition.)

Proof: Let a_i denote the position of the i th available item. We must show that $a_i - a_j = 9$ for some i and j . Consider the numbers

a_1, a_2, \dots, a_{45}

and

$a_1+9, a_2+9, \dots, a_{45}+9$

There are a total of 90 numbers in those sets. But there are only 80 items in the inventory list. Since numbers in each set are assumed to be unique, it must be the case where some number in the first set is identical to some number in the second set. Therefore, $a_i - a_j = 9$ for some i and j , as desired. QED.

- **Generalized Pigeonhole Principle**

A generalization of the pigeonhole principle states that if n pigeons fly into m pigeonholes and, for some positive integer k , $k < n/m$, then at least one pigeonhole contains $k + 1$ or more pigeons.

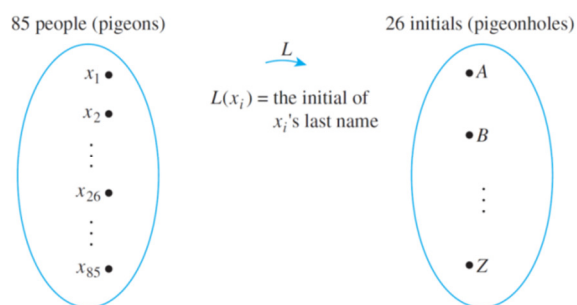
Generalized Pigeonhole Principle

For any function f from a finite set X with n elements to a finite set Y with m elements and for any positive integer k , if $k < n/m$, then there is some $y \in Y$ such that y is the image of at least $k + 1$ distinct elements of X .

- Example: [9.4.5, p. 560]

Show how the generalized pigeonhole principle implies that in a group of 85 people, at least 4 must have the same last initial.

Solution: $\frac{85}{26} = 3$, therefore by the generalized Pigeonhole Principle, there must be at least 4 people who have the same last initial.



More Examples

1. Among any group of 367 people, there must be at least two with the same birthday.
2. Among any group of 13 people, there must be at least two with the same birth-month.
3. In any group of 27 English words, there must be at least two that begin with the same letter.
4. How many students must be in a class to guarantee that at least two students receive the same score on the final exam, if the exam is graded on a scale from 0 to 100 points?
5. Show that for every integer n there is a multiple of n that has only 0s and 1s in its decimal expansion.
6. How many numbers must be selected from the set $\{1, 2, 3, 4, 5, 6\}$ to guarantee that at least one pair of these numbers add up to 7?

5 Binomial Theorem

- In algebra a sum of two terms, such as $a + b$, is called a binomial.
- The Binomial Theorem gives a formula for the coefficients in the expansion of $(a + b)^n$. Since

$$(a + b)^n = \underbrace{(a + b) \cdot (a + b) \cdot \cdots \cdot (a + b)}_{n \text{ of them}}$$

the expansion results from selecting either a or b from each of the n factors (and multiplying the selections together and summing all such products obtained).

For example, in the expansion of $(a + b)^3$, we select either a or b for the first factor, a or b again for the second factor, and do the same again for the third factor. For example, if we select a for all three factors, we obtain the term 'aaa', while if we select a for the first factor, b for the second factor, and a for the third factor, we obtain the term 'aba'.

Then the expansion is the collection of all possible terms (and sum the same terms).

$$\begin{aligned}(a + b)^3 &= (a + b) \cdot (a + b) \cdot (a + b) \\ &= aaa + aab + aba + abb + baa + bab + bba + bbb \\ &= a^3 + a^2b + a^2b + ab^2 + a^2b + ab^2 + ab^2 + b^3 \\ &= a^3 + 3a^2b + 3ab^2 + b^3\end{aligned}$$

- The **binomial theorem** gives an expression for the powers of a binomial $(a + b)^n$, for each positive integer n and all real numbers a and b .

Theorem 9.7.2 Binomial Theorem

Given any real numbers a and b and any nonnegative integer n ,

$$\begin{aligned}(a + b)^n &= \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k \\ &= a^n + \binom{n}{1} a^{n-1} b^1 + \binom{n}{2} a^{n-2} b^2 + \cdots + \binom{n}{n-1} a^1 b^{n-1} + b^n.\end{aligned}$$

Or another way to express:

$$(x + a)^n = \sum_{k=0}^n \binom{n}{k} x^k a^{n-k}$$

- If n and k are nonnegative integers and $k \leq n$, then $\binom{n}{k}$ is called a **binomial coefficient** because it is one of the coefficients in the expansion of the binomial expression $(a + b)^n$.
- Examples:

$$(a+b)^5 = \sum_{k=0}^5 \binom{5}{k} a^{5-k} b^k$$

$$= a^5 + \binom{5}{1} a^{5-1} b^1 + \binom{5}{2} a^{5-2} b^2 + \binom{5}{3} a^{5-3} b^3 + \binom{5}{4} a^{5-4} b^4 + b^5$$

$$= a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

$$(x-4y)^4 = \sum_{k=0}^4 \binom{4}{k} x^{4-k} (-4y)^k$$

$$= x^4 + \binom{4}{1} x^{4-1} (-4y)^1 + \binom{4}{2} x^{4-2} (-4y)^2 + \binom{4}{3} x^{4-3} (-4y)^3 + (-4y)^4$$

$$= x^4 + 4x^3(-4y) + 6x^2(16y^2) + 4x^1(-64y^3) + (256y^4)$$

Find the coefficient of $x^2y^3z^4$ in the expansion of $(x+y+z)^9$.

ANSWER: We have $\binom{9}{2}$ ways for x. After the selections for x is done, we have $\binom{7}{3}$ ways for y. Therefore by multiplying them, we know there are $\binom{9}{2} \cdot \binom{7}{3} = \frac{9!}{2! \cdot 7!} \cdot \frac{7!}{3! \cdot 4!} = \frac{9!}{2! \cdot 3! \cdot 4!} = 1260$ ways in total that make $x^2y^3z^4$. Therefore, the coefficient is 1260.

- We can write the binomial coefficients in a triangular form known as Pascal's triangle:

$$\begin{array}{ccccccc} & & 1 & & & & \\ & 1 & & 1 & & & \\ & 1 & 2 & 1 & & & \\ & 1 & 3 & 3 & 1 & & \\ 1 & 4 & 6 & 4 & 1 & & \\ 1 & 5 & 10 & 10 & 5 & 1 & \end{array}$$

This is based on Pascal's formula:

Theorem 9.7.1 Pascal's Formula

Let n and r be positive integers and suppose $r \leq n$. Then

$$\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}.$$

$\begin{smallmatrix} r \\ n \end{smallmatrix}$	0	1	2	3	4	5	...	$r-1$	r	...
0	1						
1	1	1					
2	1	2	1				
3	1	3	3	1			
4	1	4	6	4	1		
5	1	5	10	10	5	1	
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots		\vdots	\vdots	\vdots
n	$\binom{n}{0}$	$\binom{n}{1}$	$\binom{n}{2}$	$\binom{n}{3}$	$\binom{n}{4}$	$\binom{n}{5}$...	$\binom{n}{r-1}$	$\binom{n}{r}$...
$n+1$	$\binom{n+1}{0}$	$\binom{n+1}{1}$	$\binom{n+1}{2}$	$\binom{n+1}{3}$	$\binom{n+1}{4}$	$\binom{n+1}{5}$...	$\binom{n+1}{r}$	$\binom{n+1}{r+1}$...
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