1. Basics

- There are various kinds of relations between mathematical objects – e.g. $+$, $/$, $x^y$ (exponent), $\equiv$, $\neq$, $\geq$, $\land$, $\sim$, $\rightarrow$, $\equiv$, $\cap$
- Formal definition of (Binary) Relation:

> So, $xRy$ means $(x,y) \in R$.

- [Example 8.1.1, p. 442]: Define a relation $L$ from $R$ (real numbers) to $R$ as follows:

For all real numbers $x$ and $y$, $x \, L \, y \iff x < y$.

  a. Is 57 $L$ 53?
  b. Is $(-17) \, L \, (-14)$?
  c. Is 143 $L$ 143?
  d. Is $(-35) \, L \, 1$?

- N-ary Relations – A relation defined on several sets.

> Example: A simple database

Define a quaternary relation $R$ on $A_1 \times A_2 \times A_3 \times A_4$ as follows:

$(a_1, a_2, a_3, a_4) \in R \iff$ a patient with patient ID number $a_1$, named $a_2$, was admitted on date $a_3$, with primary diagnosis $a_4$.

Example instances/tuples:

- (011985, John Schmidt, 020710, asthma)
- (574329, Tak Kurosawa, 0114910, pneumonia)
- (466581, Mary Lazars, 0103910, appendicitis)

2. Reflexivity, Symmetry, Transitivity

- Important properties of general relations:
Informal definitions:

- **Reflexive**: Each element is related to itself.
- **Symmetric**: If any one element is related to any other element, then the second element is related to the first.
- **Transitive**: If any one element is related to a second and that second element is related to a third, then the first element is related to the third.

[Definitions for Non-relation]

1. R is not reflexive ⇔ there is an element x in A such that x R x [that is, such that (x, x) \( \notin \) R].
2. R is not symmetric ⇔ there are elements x and y in A such that x R y but y R x [that is, such that (x, y) \( \in \) R but (y, x) \( \notin \) R].
3. R is not transitive ⇔ there are elements x, y and z in A such that x R y and y R z but x R z [that is, such that (x, y) \( \in \) R and (y, z) \( \in \) R but (x, z) \( \notin \) R].

Examples:

- **[8.2.1, p. 451]** Let A = \{0, 1, 2, 3\} and define relations R, S, and T on A as follows:

  - \( R = \{(0, 0), (0, 1), (0, 3), (1, 0), (1, 1), (2, 2), (3, 0), (3, 3)\}, \)
  - \( S = \{(0, 0), (0, 2), (0, 3), (2, 3)\}, \)
  - \( T = \{(0, 1), (2, 3)\}. \)

  a. Is R reflexive? symmetric? transitive?
  b. Is S reflexive? symmetric? transitive?
  c. Is T reflexive? symmetric? transitive?

- **[8.2.3, p. 454]** Define a relation R on \( \mathbb{R} \) (the set of all real numbers) as follows:

  For all x, y ∈ \( \mathbb{R} \), x R y ⇔ x < y.


- **[8.2.4, p. 455]** Define a relation T on \( \mathbb{Z} \) (the set of all integers) as follows:

  For all integers m and n, m T n ⇔ 3 \mid (m − n).

  This relation is called congruence modulo 3.

  Is T Reflexive? Symmetric? Transitive?

Transitive Closure of a relation
Example [8.2.5, p. 457]: Let $A = \{0, 1, 2, 3\}$ and consider the relation $R$ defined on $A$ as follows:

$$R = \{(0, 1), (1, 2), (2, 3)\}.$$ 

Find the transitive closure of $R$.

**ANSWER:** \{(0, 1), (0, 2), (0, 3), (1, 2), (1, 3), (2, 3)\}.

### 4. Equivalence Relations

- A relation on a set that satisfies the three properties of reflexivity, symmetry, and transitivity is called an **equivalence relation**.

**Example:**

✓ Consider the relation $R$ on a set $\{1,2,3,4,5\}$.

$$R = \{(1,1), (1,3), (1,5), (2,2), (2,4), (3,1), (3,3), (3,5), (4,2), (4,4), (5,1), (5,3), (5,5)\}$$

is an equivalence relation because:

- $R$ is reflexive because $(1,1), (2,2), (3,3), (4,4), (5,5)$ are in $R$.
- $R$ is symmetric because whenever $(x,y)$ is in $R$, $(y,x)$ is in $R$ as well.
- $R$ is transitive because whenever $(x,y)$ and $(y,z)$ are in $R$, $(x,z)$ is in $R$ as well.

✓ Consider the relation $R$ on a set $\{1,2,3,4\}$.

$$R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)\}$$

is NOT an equivalence relation because $R$ is not symmetric.

### 5. Equivalence Classes

- “In mathematics, when the elements of some set $S$ have a notion of equivalence (formalized as an equivalence relation) defined on them, then one may naturally **split the set $S$ into equivalence classes**. These equivalence classes are constructed so that elements $a$ and $b$ belong to the same equivalence class if and only if $a$ and $b$ are equivalent.” [Wikipedia]
**Example** [8.3.5, p. 465]: Let \( A = \{0, 1, 2, 3, 4\} \) and define a relation \( R \) on \( A \) as follows:

\[
R = \{(0, 0), (0, 4), (1, 1), (1, 3), (2, 2), (3, 1), (3, 3), (4, 0), (4, 4)\}.
\]

The directed graph for \( R \) is as shown below. As can be seen by inspection, \( R \) is an equivalence relation on \( A \). Find the distinct equivalence classes of \( R \).

ANSWER: First find the equivalence class of every element of \( A \).

- \([0] = \{x \in A \mid x R 0\} = \{0, 4\}\)
- \([1] = \{x \in A \mid x R 1\} = \{1, 3\}\)
- \([2] = \{x \in A \mid x R 2\} = \{2\}\)
- \([3] = \{x \in A \mid x R 3\} = \{1, 3\}\)
- \([4] = \{x \in A \mid x R 4\} = \{0, 4\}\)

Note that \([0] = [4]\) and \([1] = [3]\). Thus the distinct equivalence classes of the relation are \([0, 4]\), \([1, 3]\), and \([2]\).

**Exercises:**
1. The relation \( R \) on a set \( \{1,2,3,4,5\} \).

\[
R = \{(1,1), (1,3), (1,5), (2,2), (2,4), (3,1), (3,3), (3,5), (4,2), (4,4), (5,1), (5,3), (5,5)\}
\]

is an equivalence relation (as shown in the previous examples). First find the equivalence classes.

2. Let \( X = \{1,2,3,...,10\} \). Define \( xRy \) to mean that 3 divides \( x-y \).

We can readily verify that \( T \) is reflexive, symmetric and transitive (thus \( R \) is an equivalent relation).

Let us determine the members of the equivalence classes. The equivalence class \([1]\) consists of all \( x \) with \( xR1 \), thus

\[
[1] = \{x \in X \mid 3 \text{ divides } x-1\} = \{1,4,7,10\}.
\]

Find all other equivalent classes.

4. **[§8.5] Partial Order Relations**
• **Antisymmetric relation** -- when symmetric elements are NOT in the relation.

A relation $R$ on a set $X$ is called *antisymmetric* if for all $x, y \in X$, if $(x, y) \in R$ then $(y, x) \notin R$.

**Example:** The relation $R$ on a set $\{1, 2, 3, 4\}$, and a relation $R$ defined over $X$ as $(x, y) \in R$ if $x \leq y$:

$R = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\}$

is antisymmetric because for all $x, y$, if $(x, y) \in R$ and $x \neq y$, then $(y, x) \notin R$.

• **Partial order relation** – VERY IMPORTANT in Computer Science (related to data structures such as trees, graphs).

Two fundamental partial order relations are the “less than or equal to ($\leq$)” relation on a set of real numbers and the “subset ($\subseteq$)” relation on a set of sets.

• **Example** [8.5.4, p. 501] Another useful partial order relation is the “divides” relation.

Let $|$ be the “divides” relation on a set $A$ of positive integers. That is, for all $a, b \in A$,

$$a \mid b \iff b = ka$$

for some integer $k$.

Prove that $|$ is a partial order relation on $A$.

**ANSWER:** is shown on the textbook, but we will work it out in the class.