[Ch 7] Functions

1. Intro to Relation

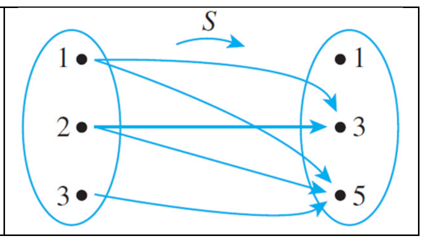
- There are various kinds of **relations** between mathematical objects – e.g. +, /, x^y (exponent), =, ≠, ≥, ∧, ~, →, ≡, ∩
- Formal definition of Relation:

> **• Definition**
>
> Let A and B be sets. A **relation** R **from** A **to** B is a subset of $A \times B$. Given an ordered pair (x, y) in $A \times B$, x **is related to** y **by** R, written $x\ R\ y$, if, and only if, (x, y) is in R. The set A is called the domain of R and the set B is called its co-domain.

So, **xRy** means **(x,y)** ∈ **R.**

- [Example 1.1.3, p. 16]: Let A = {1, 2, 3} and B = {1, 3, 5} and define relations S from A to B as follows:

  For all (x, y) ∈ A x B, (x, y) ∈ S means that x < y (thus S is a 'less than' relation).

Then we have the following for S: {(1,3), (1,5), (2,3), (2,5), (3,5)}	

- Exercise [§1.3, #6]:

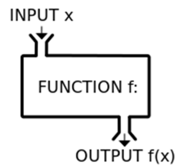
  Define a relation R from R to R as follows:

  For all (x, y) ∈ R × R, (x, y) ∈ R means that $y = x^2$.

  a. Is (2, 4) ∈ R? Is (4, 2) ∈ R? Is (–3) R 9? Is 9 R (–3)?
  b. Draw the graph of R in the Cartesian plane.

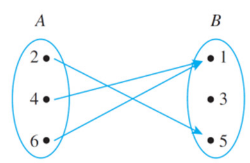
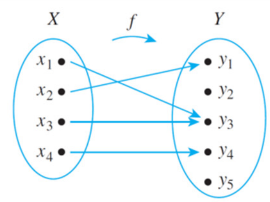
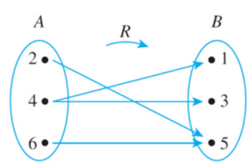
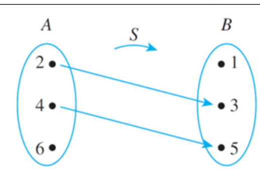
2. Functions – Basic Concepts and Definitions

- "In mathematics, a **function** is a relation between a set of inputs and a set of permissible outputs with the property that **EACH input** is related to **EXACTLY ONE** output." [Wikipedia].

A function **f** takes an input x, and returns a single output f(x).	

- A little more formal definition [§1.3]:

> • **Definition**
>
> A **function** F **from a set** A **to a set** B is a relation with domain A and co-domain B that satisfies the following two properties:
>
> 1. For every element x in A, there is an element y in B such that $(x, y) \in F$.
> 2. For all elements x in A and y and z in B,
>
> $$\text{if} \quad (x, y) \in F \text{ and } (x, z) \in F, \quad \text{then} \quad y = z.$$

Function		
Non-Function		

- Notation:
  1. **b = F(a)** where a ∈ A and b ∈ B.
  2. **F: A → B**

- <u>Example</u> [1.3.5, p. 18]: For all (x, y) ∈ R × R,

  - (x, y) ∈ L means that y = x - 1. Is L a function?
  - (x, y) ∈ C means that $x^2 + y^2 = 1$. Is C a function?

- Example functions
  - Square-root function
  - Power function
  - Constant function
  - Successor function (for sequences)

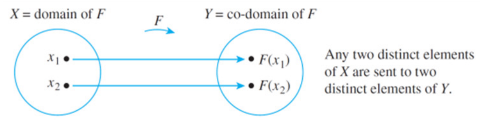
- **Various special functions:**
  1. Identity function … I = f(I)
  2. Binary function … B = f(A) where B = {True, False}

3. One-to-One and Onto, Inverse Functions

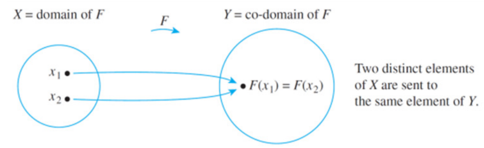
- Two important properties of functions:

1. **One-to-One (Injection)** … No two elements in the domain are related to the same element in the co-domain.

> **• Definition**
>
> Let F be a function from a set X to a set Y. F is **one-to-one** (or **injective**) if, and only if, for all elements x_1 and x_2 in X,
>
> $$\text{if } F(x_1) = F(x_2), \text{ then } x_1 = x_2,$$
>
> or, equivalently, $\quad \text{if } x_1 \neq x_2, \text{ then } F(x_1) \neq F(x_2).$
>
> Symbolically,
>
> $$F: X \to Y \text{ is one-to-one} \Leftrightarrow \forall x_1, x_2 \in X, \text{ if } F(x_1) = F(x_2) \text{ then } x_1 = x_2.$$



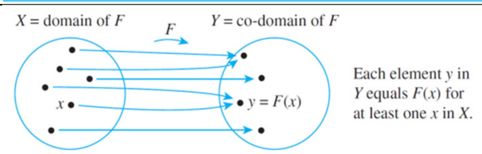
One-to-One



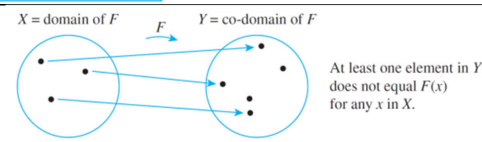
NOT One-to-One

2. **Onto (Surjection)** … Every element in the co-domain has a related element in the domain.

> **• Definition**
>
> Let F be a function from a set X to a set Y. F is **onto** (or **surjective**) if, and only if, given any element y in Y, it is possible to find an element x in X with the property that $y = F(x)$.
>
> Symbolically:
>
> $$F: X \to Y \text{ is onto} \Leftrightarrow \forall y \in Y, \exists x \in X \text{ such that } F(x) = y.$$



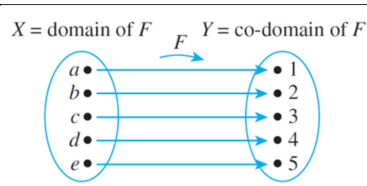
Onto



NOT Onto

3. **One-to-One and Onto (bijection; One-to-One Correspondence)** … Every element in the domain has a unique corresponding element in the co-domain.

> **• Definition**
>
> A **one-to-one correspondence** (or **bijection**) from a set X to a set Y is a function $F: X \to Y$ that is both one-to-one and onto.



- <u>Examples</u>:
  1. [7.2.2, p. 399] Define $f: R \to R$ and $g: Z \to Z$ by the rules

     $f(x) = 4x - 1$ for all $x \in R$, and $g(n) = n^2$ for all $n \in Z$.

     a. Is f one-to-one? Prove or give a counterexample.
     b. Is g one-to-one? Prove or give a counterexample.

  2. [7.2.5, p. 403] Define $f: R \to R$ and $h: Z \to Z$ by the rules

f (x) = 4x − 1 for all x ∈ R, and h(n) = 4n − 1 for all n ∈ Z.

a. Is f onto? Prove or give a counterexample.
b. Is h onto? Prove or give a counterexample.

3. [7.2.9, p. 409] Let T be the set of all finite strings of x's and y's. Define g: T → T by the rule:

For all strings s ∈ T , g(s) = the string obtained by writing the characters of s in reverse order.

a. Is g a one-to-one correspondence from T to itself?

4. Inverse Function

- If a function F is a one-to-one correspondence from a set X to a set Y, there is a function from Y to X that "undoes" the action of F.

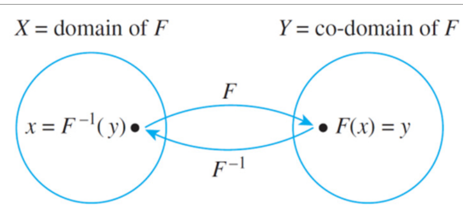
Theorem 7.2.2

Suppose $F: X \to Y$ is a one-to-one correspondence; that is, suppose F is one-to-one and onto. Then there is a function $\boldsymbol{F^{-1}: Y \to X}$ that is defined as follows:
Given any element y in Y,

$$F^{-1}(y) = \text{that unique element } x \text{ in } X \text{ such that } F(x) \text{ equals } y.$$

In other words,

$$F^{-1}(y) = x \quad \Leftrightarrow \quad y = F(x).$$



- Example [7.2.13, p. 412]: The function f : R → R defined by the formula

  f (x) = 4x − 1 for all real numbers x

  was shown to be one-to-one in Example 7.2.2 and onto in Example 7.2.5. Find its inverse function.