# [Ch 6] Set Theory

## 1. Basic Concepts and Definitions

### 1) Basics

- **Element**: $= \{\boldsymbol{x} \in \boldsymbol{S} |\ \boldsymbol{P}(\boldsymbol{x})\}$ ; A is a set consisting of elements x which is in a/another set S such that P(x) is true.
- **Empty set**: notated { } (or ∅)

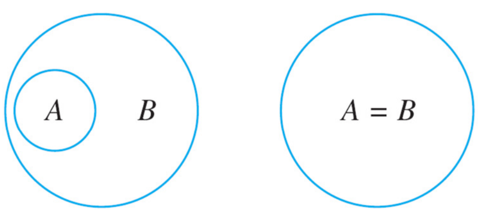
- **Subset**: $\boldsymbol{A} \subseteq \boldsymbol{B}$ ; A is a subset of B. This also implies **∀x, if x ∈ A then x ∈ B.**
  - Example: $A = \{3, 4, 5\},\ B = \{x \in Z |\ x \geq 2\}, C = \{3,4,5\}$

    Then we have relations (a partial list): $A \subseteq B, C \subseteq B, A \subseteq C, C \subseteq A$

- **Proper subset**: $\boldsymbol{A} \subset \boldsymbol{B}$ ; (1) A is a subset of B, and (2) there is at least one element in B that is not in A.
  - Example: $A = \{3, 4, 5\},\ B = \{x \in Z |\ x \geq 2\}, C = \{3,4,5\}$

    Then followings are true (a partial list): $A \subset B, C \subset B, A \not\subset C, C \not\subset A$

- Set **equality**: A = B ; this happens if and only if $\boldsymbol{A} \subseteq \boldsymbol{B}$ ***and*** $\boldsymbol{B} \supseteq \boldsymbol{A}$**.**




$\boldsymbol{A} \subseteq \boldsymbol{B}$ $\quad$ $\boldsymbol{A} = \boldsymbol{B}$

  - Example: [Example 6.1.3 Set Equality, p. 339]

  Define sets A and B as follows:

  A = {m ∈ Z | m = 2a for some integer a}
  B = {n ∈ Z | n = 2b − 2 for some integer b}

  Is A = B?

  **Solution:** Yes. To prove this, both subset relations A ⊆ B and B ⊆ A must be proved.

  a. Part 1, Proof That A ⊆ B:

     Suppose x is a particular but arbitrarily chosen element of A. [We must show that x ∈ B. By definition of B, this means we must show that x = 2·(some integer)−2.]

By definition of A, there is an integer a such that x = 2a. [Given that x = 2a, can x also be expressed as 2·(some integer)–2? I.e., is there an integer, say b, such that 2a = 2b – 2? Solve for b to obtain b = (2a + 2)/2 = a + 1. Check to see if this works.]

Let b = a + 1.
[First check that b is an integer.] Then b is an integer because it is a sum of integers.
[Then check that x= 2b – 2.] Also 2b – 2 = 2(a + 1) – 2 = 2a + 2 – 2 = 2a = x,

Thus, by definition of B, x is an element of B [which is what was to be shown].

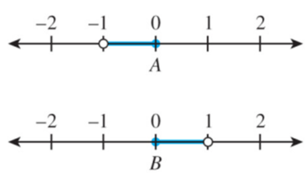
b. Part 2, Proof That $B \subseteq A$:

Will work on this part in the class.

**2) Intervals**

- To represent an interval on a one-dimensional space, square brackets ([ ]) and parentheses ( ) are used, where [ ] denotes **inclusion** and () denotes **exclusion**.

$A = (-1,\ 0] = \{x \in \mathbf{R} \mid -1 < x \le 0\}$ and $B = [0,\ 1) = \{x \in \mathbf{R} \mid 0 \le x < 1\}$.




## 2. Operations on sets

**• Definition**

Let $A$ and $B$ be subsets of a universal set $U$.

1. The **union** of $A$ and $B$, denoted $\boldsymbol{A \cup B}$, is the set of all elements that are in at least one of $A$ or $B$.

2. The **intersection** of $A$ and $B$, denoted $\boldsymbol{A \cap B}$, is the set of all elements that are common to both $A$ and $B$.

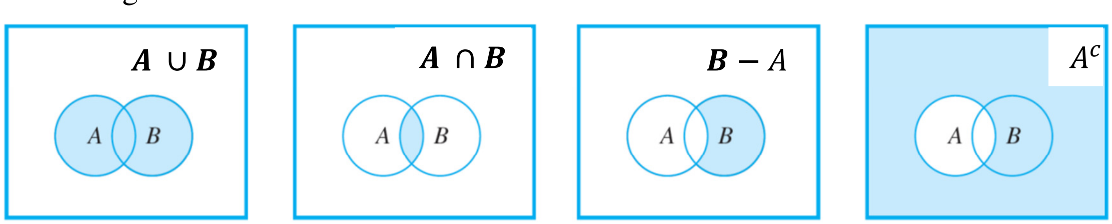
3. The **difference** of $B$ minus $A$ (or **relative complement** of $A$ in $B$), denoted $\boldsymbol{B - A}$, is the set of all elements that are in $B$ and not $A$.

4. The **complement** of $A$, denoted $\boldsymbol{A^c}$, is the set of all elements in $U$ that are not in $A$.

Symbolically:

$$A \cup B = \{x \in U \mid x \in A \text{ or } x \in B\},$$
$$A \cap B = \{x \in U \mid x \in A \text{ and } x \in B\},$$
$$B - A = \{x \in U \mid x \in B \text{ and } x \notin A\},$$
$$A^c = \{x \in U \mid x \notin A\}.$$

Venn Diagrams



- Example [Example 6.1.5]

  Let the universal set be the set U = {a, b, c, d, e, f, g} and let A = {a, c, e, g} and B = {d, e, f, g}. Find A ∪ B, A ∩ B, B – A, and $A^c$.

  Solution: A ∪ B = {a, c, d, e, f, g}, A ∩ B = {e, g}, B – A = {d, f}, $A^c$ = {b, d, f}

  ★ Notice NO DUPLICATES in union and intersection sets.

- Exercise 1: [Section 6.1, Exercise #10, p. 350]

  Let the universal set be the set R of all real numbers and let A = {x ∈ R| 0 < x ≤ 2}, B = {x ∈ R| 1 ≤ x < 4}, and C = {x ∈ R| 3 ≤ x < 9}. Find each of the following:

  a. A ∪ B
  b. A ∩ B
  c. $A^c$
  d. A ∪ C
  e. A ∩ C
  f. $B^c$

- Exercise 2: [Section 6.1, Exercise #3, p. 350, modified]

  Let sets R and S be defined as follows:

  R = {x ∈ Z | x is divisible by 2}
  S = {y ∈ Z | y is divisible by 3}

  a. Characterize the set R ∩ S
  b. Characterize the set R ∪ S
  c. Characterize the set $(R \cup S)^c$

**More Concepts**

- **Disjoint sets**: $\boldsymbol{A} \cap \boldsymbol{B} = \{\}$
- **Mutually disjoint sets:**

  *Definition*: Sets $A_1$, $A_2$,.. $A_n$ are mutually disjoint (i.e., non-overlapping) if and only if for any two sets $A_i$ and $A_1$, $A_i \cap A_j = \emptyset$ (empty set).

  - Example: B1 = {2, 4, 6}, B2 = {3, 7}, and B3 = {0, 5}. B1, B2, B3 are mutually disjoint.

- **Partition of a set**:

  *Definition*: Sets $A_1$, $A_2$,.. $A_n$ are a partition of a set B if and only if:

  - B is a union of all the $A_i$, that is, $B = \bigcup_{i=1}^{n} A_i$, and
  - $A_1$, $A_2$,.. $A_n$ are mutually disjoint.

- Examples:

    1. Let A = {1, 2, 3, 4, 5, 6}, A1 = {1, 6}, A2 = {3}, and A3 = {2, 4, 5}. The set of sets {A1, A2, A3} is a partition of A.
    2. Let Z be the set of all integers and let

        - T0 = {n ∈ Z| n = 3k, for some integer k},
        - T1 = {n ∈ Z| n = 3k + 1, for some integer k}, and
        - T2 = {n ∈ Z| n = 3k + 2, for some integer k}.

        The set of sets {T0, T1, T2} is a partition of Z.

- Exercise [Section 6.1, #28, p. 351]

    Let E be the set of all even integers and O the set of all odd integers. Is {E, O} a partition of Z, the set of all integers? Explain your answer.

- **Power set** of A (denoted ℘(A)) is the set of al subsets of A.
    - Example: Find the power set of the set {1, 2, 3}.

        Answer: {{}, {1}, {2}, {3}, {1,2}, {2,3}, {1,3}, {1,2,3}}

- **Cartesian Products** of n sets $A_1$,..$A_n$, denoted $\boldsymbol{A_1} \times \boldsymbol{A_2} \times \ldots \times \boldsymbol{A_n}$, is the set of all ordered n-tuples $\{\boldsymbol{a_1}, \boldsymbol{a_2}, \ldots, \boldsymbol{a_n}\}$ where $\boldsymbol{a_1} \in \boldsymbol{A_1}$, $\boldsymbol{a_2} \in \boldsymbol{A_2}$ and so on.
  In particular, $\boldsymbol{A_1} \times \boldsymbol{A_2} = \{(\boldsymbol{a_1}, \boldsymbol{a_2}) | \boldsymbol{a_1} \in \boldsymbol{A_1} \ \boldsymbol{and} \ \boldsymbol{a_2} \in \boldsymbol{A_2}\}$.

    - Exercise [Section 6.1, #31, p. 351]
      Suppose A = {1, 2} and B = {2, 3}. Find each of the following:

        - ℘ (A ∩ B)
        - ℘ (A)
        - ℘ (A ∪ B)
        - ℘ (A × B)

## 3. Set Properties

- Some basic properties of sets which involve union, intersection, complement and difference.

**Theorem 6.2.1 Some Subset Relations**

1. *Inclusion of Intersection:* For all sets $A$ and $B$,

$$(a)\ A \cap B \subseteq A \quad \text{and} \quad (b)\ A \cap B \subseteq B.$$

2. *Inclusion in Union:* For all sets $A$ and $B$,

$$(a)\ A \subseteq A \cup B \quad \text{and} \quad (b)\ B \subseteq A \cup B.$$

3. *Transitive Property of Subsets:* For all sets $A$, $B$, and $C$,

$$\text{if } A \subseteq B \text{ and } B \subseteq C, \text{ then } A \subseteq C.$$

**Procedural Versions of Set Definitions**

Let $X$ and $Y$ be subsets of a universal set $U$ and suppose $x$ and $y$ are elements of $U$.

1. $x \in X \cup Y \quad \Leftrightarrow \quad x \in X$ or $x \in Y$
2. $x \in X \cap Y \quad \Leftrightarrow \quad x \in X$ and $x \in Y$
3. $x \in X - Y \quad \Leftrightarrow \quad x \in X$ and $x \notin Y$
4. $x \in X^c \quad \Leftrightarrow \quad x \notin X$
5. $(x, y) \in X \times Y \quad \Leftrightarrow \quad x \in X$ and $y \in Y$

- Example: [Example 6.2.1, p. 353] Prove Theorem 6.2.1(1)(a): For all sets A and B, A ∩ B ⊆ A.

  Proof: [Skeleton only] To Show: A ∩ B ⊆ A.

  **To prove that A ∩ B ⊆ A, you must show that ∀x, if x ∈ A ∩ B then x ∈ A.** 🡐 from the Procedural version of Set Definitions above.
  Steps:

  - You suppose x is an element in A ∩ B, and then
  - You show that x is in A.

  To say that x is in A ∩ B means that x is in A and x is in B.
  This allows you to complete the proof by deducing that, in particular, x is in A, as was to be shown.
  Note that this deduction is just a special case of the valid argument form

  p ∧ q
  ∴ p.

## Set Identities

- Identity relations on sets

**Theorem 6.2.2 Set Identities**

Let all sets referred to below be subsets of a universal set $U$.

1. *Commutative Laws:* For all sets $A$ and $B$,

$$\text{(a) } A \cup B = B \cup A \quad \text{and} \quad \text{(b) } A \cap B = B \cap A.$$

2. *Associative Laws:* For all sets $A$, $B$, and $C$,

$$\text{(a) } (A \cup B) \cup C = A \cup (B \cup C) \quad \text{and}$$
$$\text{(b) } (A \cap B) \cap C = A \cap (B \cap C).$$

3. *Distributive Laws:* For all sets, $A$, $B$, and $C$,

$$\text{(a) } A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \quad \text{and}$$
$$\text{(b) } A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

4. *Identity Laws:* For all sets $A$,

$$\text{(a) } A \cup \emptyset = A \quad \text{and} \quad \text{(b) } A \cap U = A.$$

5. *Complement Laws:*

$$\text{(a) } A \cup A^c = U \quad \text{and} \quad \text{(b) } A \cap A^c = \emptyset.$$

6. *Double Complement Law:* For all sets $A$,

$$(A^c)^c = A.$$

7. *Idempotent Laws:* For all sets $A$,

$$\text{(a) } A \cup A = A \quad \text{and} \quad \text{(b) } A \cap A = A.$$

8. *Universal Bound Laws:* For all sets $A$,

$$\text{(a) } A \cup U = U \quad \text{and} \quad \text{(b) } A \cap \emptyset = \emptyset.$$

9. *De Morgan's Laws:* For all sets $A$ and $B$,

$$\text{(a) } (A \cup B)^c = A^c \cap B^c \quad \text{and} \quad \text{(b) } (A \cap B)^c = A^c \cup B^c.$$

10. *Absorption Laws:* For all sets $A$ and $B$,

$$\text{(a) } A \cup (A \cap B) = A \quad \text{and} \quad \text{(b) } A \cap (A \cup B) = A.$$

11. *Complements of $U$ and $\emptyset$:*

$$\text{(a) } U^c = \emptyset \quad \text{and} \quad \text{(b) } \emptyset^c = U.$$

12. *Set Difference Law:* For all sets $A$ and $B$,

$$A - B = A \cap B^c.$$

- **Proving Set Identities (procedurally; by element membership)**

  To prove two sets are equal, we must show both directions of the subset relation:

**Basic Method for Proving That Sets Are Equal**

Let sets $X$ and $Y$ be given. To prove that $X = Y$:

1. Prove that $X \subseteq Y$.
2. Prove that $Y \subseteq X$.

Also again, use the **procedural version of the set** definitions and show the membership of the elements**.**

- Example 1: [Example 6.2.3 Proof of DeMorgan's Law for Sets, p. 359]

  Prove (true) that for all sets A and B, $(A \cup B)^c = A^c \cap B^c$.

  Proof: [Skeleton only] We must show that $(A \cup B)^c \subseteq A^c \cap B^c$ and that $A^c \cap B^c \subseteq (A \cup B)^c$.

  To show the first containment means to show that **$\forall x$, if $x \in (A \cup B)^c$ then $x \in A^c \cap B^c$**.
  And to show the second containment means to show that **$\forall x$, if $x \in A^c \cap B^c$ then $x \in (A \cup B)^c$**.

  Since each of these statements is universal and conditional, for the first containment, you suppose $x \in (A \cup B)^c$, and then you show that $x \in A^c \cap B^c$.
  And for the second containment, you suppose $x \in A^c \cap B^c$, and then you show that $x \in (A \cup B)^c$.

  To fill in the steps of these arguments, you use the procedural versions of the definitions of complement, union, and intersection, and at crucial points you use De Morgan's laws of logic. [The entire proof is presented on p. 360-361].

- Example 2: [Example 6.3.1 Finding a Counterexample, p. 367]

  Is the following set property true? -- For all sets A, B, and C, $(A - B) \cup (B - C) = A - C$.

  Proof: By counterexample.
  Let $A = \{1, 2, 3, 4, 5\}$, $B = \{2, 3, 5, 6\}$ and $C = \{4, 5, 6, 7\}$. Then

  $A - B = \{1, 4\}$, $B - C = \{2, 3\}$, $A - C = \{1, 2, 3\}$

  Hence

  $(A - B) \cup (B - C) = \{1, 4\} \cup \{2, 3\} = \{1, 2, 3, 4\}$, whereas $A - C = \{1, 2, 3\}$.

  Since $\{1, 2, 3, 4, 5\} \neq \{1, 2, 3\}$, we have that $(A - B) \cup (B - C) \neq A - C$.

- **Exercises**:

1. [Section 6.2, Exercise #3, p. 365]

   The following is a proof that for all sets A, B, and C, if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$. Fill in the blanks.

   … Will work on this in the class, but you can find an answer at the back of the textbook.

Proof: Suppose A, B, and C are sets and $A \subseteq B$ and $B \subseteq C$. To show that $A \subseteq C$, we must show that every element in __(a)__ is in __(b)__. But given any element in A, that element is in __(c)__ (because $A \subseteq B$), and so that element is also in __(d)__ (because __(e)__). Hence $A \subseteq C$.

2. [Section 6.2, Exercise #5, p. 365]

   Prove that for all sets A and B, $(B - A) = B \cap A^c$.

   Proof: ... Will work on this in the class, but you can find an answer at the back of the textbook.

3. [Section 6.3, Exercise #6, P. 372] Prove the statement if that is true, or find a counterexample if false. Assume all sets are subsets of a universal set U.

   For all sets A, B, and C, $A \cap (A \cup B) = A$.

   Proof: ... Will work on this in the class, but you can find an answer at the back of the textbook.

- **Proving Set Identities Algebraically**

Alternatively, we can prove set properties *algebraically* using the set identity laws.

- Example: [Example 6.3.2 Deriving a Set Difference Property, p. 371]

  Construct an algebraic proof that for all sets A, B, and C,

  $$(A \cup B) - C = (A - C) \cup (B - C).$$

  Cite a property from Theorem 6.2.2 for every step of the proof.

  Solution: Let A, B, and C be any sets. Then

  $(A \cup B) - C = (A \cup B) \cap C^c$ by the set difference law
  $= C^c \cap (A \cup B)$ by the commutative law for $\cap$
  $= (C^c \cap A) \cup (C^c \cap B)$ by the distributive law
  $= (A \cap C^c) \cup (B \cap C^c)$ by the commutative law for $\cap$
  $= (A - C) \cup (B - C)$ by the set difference law.

- Exercise: [Section 6.3, #31, p. 373]

  For all sets A and B, $A \cup (B-A) = A \cup B$.

  Proof: ... Will work on this in the class, but you can find an answer at the back of the textbook.

## 4. Boolean Algebra

1) **Correspondence between logical equivalency rules and Set properties:**

- Logical operator $\wedge$ --- Set $\cap$
- Logical operator $\vee$ --- Set $\cup$
- Logical operator ~ --- Set $^c$ (complement)
- Logical value t (a tautology) – Set U (a universal set)

- Logical value c (a contradiction) – Set ∅ (an empty set)

| Logical Equivalences | Set Properties |
|---|---|
| For all statement variables $p$, $q$, and $r$: | For all sets $A$, $B$, and $C$: |
| a. $p \vee q \equiv q \vee p$<br>b. $p \wedge q \equiv q \wedge p$ | a. $A \cup B = B \cup A$<br>b. $A \cap B = B \cap A$ |
| a. $p \wedge (q \wedge r) \equiv p \wedge (q \wedge r)$<br>b. $p \vee (q \vee r) \equiv p \vee (q \vee r)$ | a. $A \cup (B \cup C) \equiv A \cup (B \cup C)$<br>b. $A \cap (B \cap C) \equiv A \cap (B \cap C)$ |
| a. $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$<br>b. $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ | a. $A \cap (B \cup C) \equiv (A \cap B) \cup (A \cap C)$<br>b. $A \cup (B \cap C) \equiv (A \cup B) \cap (A \cup C)$ |
| a. $p \vee \mathbf{c} \equiv p$<br>b. $p \wedge \mathbf{t} \equiv p$ | a. $A \cup \emptyset = A$<br>b. $A \cap U = A$ |
| a. $p \vee \sim p \equiv \mathbf{t}$<br>b. $p \wedge \sim p \equiv \mathbf{c}$ | a. $A \cup A^c = U$<br>b. $A \cap A^c = \emptyset$ |
| $\sim(\sim p) \equiv p$ | $(A^c)^c = A$ |
| a. $p \vee p \equiv p$<br>b. $p \wedge p \equiv p$ | a. $A \cup A = A$<br>b. $A \cap A = A$ |
| a. $p \vee \mathbf{t} \equiv \mathbf{t}$<br>b. $p \wedge \mathbf{c} \equiv \mathbf{c}$ | a. $A \cup U = U$<br>b. $A \cap \emptyset = \emptyset$ |
| a. $\sim(p \vee q) \equiv \sim p \wedge \sim q$<br>b. $\sim(p \wedge q) \equiv \sim p \vee \sim q$ | a. $(A \cup B)^c = A^c \cap B^c$<br>b. $(A \cap B)^c = A^c \cup B^c$ |
| a. $p \vee (p \wedge q) \equiv p$<br>b. $p \wedge (p \vee q) \equiv p$ | a. $A \cup (A \cap B) \equiv A$<br>b. $A \cap (A \cup B) \equiv A$ |
| a. $\sim\mathbf{t} \equiv \mathbf{c}$<br>b. $\sim\mathbf{c} \equiv \mathbf{t}$ | a. $U^c = \emptyset$<br>b. $\emptyset^c = U$ |

- In fact, both are special cases of the same general structure, known as a Boolean algebra.

2) **Definition of Boolean Algebra**

- [Wikipedia, https://en.wikipedia.org/wiki/Boolean_algebra#Boolean_algebras:_the_definition ]
"In mathematics and mathematical logic, Boolean algebra is the branch of algebra in which the **values of the variables are the truth values true and false**, usually denoted 1 and 0 respectively. Instead of elementary algebra where the values of the variables are numbers, and the main operations are addition and multiplication, the **main operations of Boolean algebra are the conjunction and denoted as ∧, the disjunction or denoted as ∨, and the negation not denoted as ¬**. It is thus a formalism for describing logical relations in the same way that ordinary algebra describes numeric relations."

- [Textbook p. 375]

## • Definition: Boolean Algebra

A **Boolean algebra** is a set $B$ together with two operations, generally denoted $+$ and $\cdot$, such that for all $a$ and $b$ in $B$ both $a+b$ and $a\cdot b$ are in $B$ and the following properties hold:

1. *Commutative Laws:* For all $a$ and $b$ in $B$,

$$\text{(a) } a+b=b+a \quad \text{and} \quad \text{(b) } a\cdot b=b\cdot a.$$

2. *Associative Laws:* For all $a$, $b$, and $c$ in $B$,

$$\text{(a) } (a+b)+c=a+(b+c) \quad \text{and} \quad \text{(b) } (a\cdot b)\cdot c=a\cdot(b\cdot c).$$

3. *Distributive Laws:* For all $a$, $b$, and $c$ in $B$,

$$\text{(a) } a+(b\cdot c)=(a+b)\cdot(a+c) \quad \text{and} \quad \text{(b) } a\cdot(b+c)=(a\cdot b)+(a\cdot c).$$

4. *Identity Laws:* There exist distinct elements 0 and 1 in $B$ such that for all $a$ in $B$,

$$\text{(a) } a+0=a \quad \text{and} \quad \text{(b) } a\cdot 1=a.$$

5. *Complement Laws:* For each $a$ in $B$, there exists an element in $B$, denoted $\overline{a}$ and called the **complement** or **negation** of $a$, such that

$$\text{(a) } a+\overline{a}=1 \quad \text{and} \quad \text{(b) } a\cdot\overline{a}=0.$$

- In any Boolean algebra,
  - the complement of each element is unique; and
  - the quantities 0 and 1 are unique.

**Theorem 6.4.1 Properties of a Boolean Algebra**

Let $B$ be any Boolean algebra.

1. *Uniqueness of the Complement Law:* For all $a$ and $x$ in $B$, if $a+x=1$ and $a\cdot x=0$ then $x=\overline{a}$.

2. *Uniqueness of* 0 *and* 1: If there exists $x$ in $B$ such that $a+x=a$ for all $a$ in $B$, then $x=0$, and if there exists $y$ in $B$ such that $a\cdot y=a$ for all $a$ in $B$, then $y=1$.

3. *Double Complement Law:* For all $a\in B$, $\overline{(\overline{a})}=a$.

4. *Idempotent Law:* For all $a\in B$,

$$\text{(a) } a+a=a \quad \text{and} \quad \text{(b) } a\cdot a=a.$$

5. *Universal Bound Law:* For all $a\in B$,

$$\text{(a) } a+1=1 \quad \text{and} \quad \text{(b) } a\cdot 0=0.$$

6. *De Morgan's Laws:* For all $a$ and $b\in B$,

$$\text{(a) } \overline{a+b}=\overline{a}\cdot\overline{b} \quad \text{and} \quad \text{(b) } \overline{a\cdot b}=\overline{a}+\overline{b}.$$

7. *Absorption Laws:* For all $a$ and $b\in B$,

$$\text{(a) } (a+b)\cdot a=a \quad \text{and} \quad \text{(b) } (a\cdot b)+a=a.$$

8. *Complements of* 0 *and* 1:

$$\text{(a) } \overline{0}=1 \quad \text{and} \quad \text{(b) } \overline{1}=0.$$

- Exercise [Section 6.4, Exercise #1, p. 381]

  Assume that B is a Boolean algebra with operations + and $\cdot$. Give the reasons needed to fill in the blanks in the proofs, but do not use any parts of Theorem 6.4.1 unless they have already been proved. You may use any part of the definition of a Boolean algebra and the results of previous exercises, however.

  For all a in B, $a \cdot a = a$.

  Proof: Let a be any element of B. Then

  | | |
  |---|---|
  | $a = a \cdot 1$ | (a) |
  | $= a \cdot (a + a)$ | (b) |
  | $= (a \cdot a) + (a \cdot a)$ | (c) |
  | $= (a \cdot a) + 0$ | (d) |
  | $= a \cdot a$ | (e) |

3) **Duality principle of Boolean Algebra**

- Note that each of the paired statements can be obtained from the other by interchanging all the + and $\cdot$ signs and interchanging 1 and 0. Such interchanges transform any Boolean identity into its **dual** identity.
- It can be proved that the dual of any Boolean identity is also an identity. This fact is often called the **duality principle** for a Boolean algebra.