

## [Ch 3, 4] Logic and Proofs (2)

## 1. Valid and Invalid Arguments (§2.3, §3.4)

## 1) Basics

- An *argument* is a sequence of statements (e.g.  $s_1, s_2, \dots, s_n$ ).
- All statements in an argument, except for the final one are **premises** (or **assumptions** or **hypotheses**).
- The final statement is the **conclusion**.
- An argument is **valid** when all premises are true, the conclusion is also true. In fact, the conclusion is 'inferred' or 'deduced' from the truth of the premises.
- An argument is **invalid** when all premises are true but the conclusion is false.

To determine the validity of an argument, we go by these steps:

1. Construct a truth table for all statements involved.
2. Identify the rows in which all premises are true – “**critical rows**”.
3. If the conclusion for all critical rows are true, the argument is valid. Otherwise the argument is invalid.

e.g.

$$p \rightarrow q \vee \sim r$$

$$q \rightarrow p \wedge r$$

$$\text{Therefore } p \rightarrow r$$

| premises |     |     |          |                 |              | conclusion                    |                            |                   |
|----------|-----|-----|----------|-----------------|--------------|-------------------------------|----------------------------|-------------------|
| $p$      | $q$ | $r$ | $\sim r$ | $q \vee \sim r$ | $p \wedge r$ | $p \rightarrow q \vee \sim r$ | $q \rightarrow p \wedge r$ | $p \rightarrow r$ |
| T        | T   | T   | F        | T               | T            | T                             | T                          | T                 |
| T        | T   | F   | T        | T               | F            | T                             | F                          |                   |
| T        | F   | T   | F        | F               | T            | F                             | T                          |                   |

This row shows that an argument of this form can

|   |   |   |   |   |   |   |   |   |
|---|---|---|---|---|---|---|---|---|
| T | F | F | T | T | F | T | T | F |
| F | T | T | F | T | F | T | F |   |
| F | T | F | T | T | F | T | F |   |
| F | F | T | F | F | F | T | T | T |
| F | F | F | T | T | F | T | T | T |

have true premises and a false conclusion. Hence this form of argument is invalid.

- Exercise: [Section 2.3, Exercise #8, p. 61]  
Use truth table to determine whether the given argument form is valid or invalid.

$$p \vee q$$

$$p \rightarrow \sim q$$

$$p \rightarrow r$$

$$\therefore r$$

## 2) Modus Ponens and Modus Tollens

- An argument which consists of two premises and a conclusion is called a syllogism.
- Two forms of syllogisms:
  1. **Modus Ponens** (“Method of affirmation”)

If p then q.

p

Therefore q

e.g. All humans are mortal.  
Socrates is a human.  
Therefore, Socrates is mortal.

2. **Modus Tollens** (“Method of denying”)

If p then q.

$\sim q$

Therefore  $\sim p$

e.g. All humans are mortal.  
Zeus is not mortal.  
Therefore, Zeus is not a human.

- Exercise: [Section 2.3, #26, 27, p. 62]  
For each problem, write the logical form of each argument, and determine if the argument is valid or invalid.
  - a) If I go to the movies, I won’t finish my homework. If I don’t finish my homework, I won’t do well on the exam tomorrow.  
 $\therefore$  If I go to the movies, I won’t do well on the exam.
  - b) If this number is larger than 2, then its square is larger than 4. This number is not larger than 2.  
 $\therefore$  The square of this number is not larger than 4.

## 3) Rules of Inference

- Some general forms of logical inferences (i.e., Rules of Inference).

|                |  |                |
|----------------|--|----------------|
| Generalization | p<br>$\therefore p \vee q$ (for any statement of q)                    |                |
| Specialization | $p \wedge q$   | $p \wedge q$   |
|                | p  | q              |
| Elimination    | $p \vee q$   | $p \vee q$     |
|                | $\sim q$   | $\sim p$       |
|                | $\therefore p$   | $\therefore q$ |
| Transitivity   | $p \rightarrow q$<br>$q \rightarrow r$<br>$\therefore p \rightarrow r$ |                |
| Conjunction    | p<br>q<br>$\therefore p \wedge q$                                      |                |

|                                    |  |
|------------------------------------|--|
| Contradiction                      | $\sim p \rightarrow c$ (where c means contradiction)<br>therefore p    |
| Proof by<br>Division into<br>Cases | $p \vee q$<br>$p \rightarrow r$<br>$q \rightarrow r$<br>$\therefore r$ |

- Exercise: [Section 2.3, Exercise #41, p. 63]  
Given a set of premises and a conclusion, use the rules of inferences to deduce the conclusion from the premises. Assume all variables are statement variables.

- a.  $\sim p \vee q \rightarrow r$
- b.  $s \vee \sim q$
- c.  $\sim t$
- d.  $p \rightarrow t$
- e.  $\sim p \wedge r \rightarrow \sim s$
- f.  $\therefore \sim q$

#### 4) Formal Proof

- A formal proof of a conclusion C, given premises  $p_1, p_2, \dots, p_n$  consists of a sequence of steps, each of which applies some inference rule to premises or previously-proven statements (antecedents) to yield a new true statement (the consequent).
- A proof demonstrates that if the premises are true, then the conclusion is true.

Example: Suppose we have the following premises:

- “It is not sunny and it is cold.”
- “We will swim if only if it is sunny.”
- “If we do not swim, then we will canoe.”
- “If we canoe, then we will be home early.”

Given these premises, prove the theorem

- “We will be home early” using inference rules.

Proof: Let us adopt the following abbreviations:

sunny = “It is sunny”;  
cold = “It is cold”;  
swim = “We will swim”;  
canoe = “We will canoe”;  
early = “We will be home early”.

Then, the premises can be written as:

- (1)  $\sim \text{sunny} \wedge \text{cold}$  (2)  $\text{swim} \leftrightarrow \text{sunny}$
- (3)  $\sim \text{swim} \rightarrow \text{canoe}$  (4)  $\text{canoe} \rightarrow \text{early}$

Inference steps:

- |  |                         |
|--|-------------------------|
| 1. $\sim \text{sunny} \wedge \text{cold}$      | Premise #1.             |
| 2. $\sim \text{sunny}$                         | Simplification of 1.    |
| 3. $\text{swim} \leftrightarrow \text{sunny}$  | Premise #2.             |
| 4. $\sim \text{swim}$                          | IFF Truth table on 2,3. |
| 5. $\sim \text{swim} \rightarrow \text{canoe}$ | Premise #3.             |
| 6. canoe                                       | Modus ponens on 4,5.    |
| 7. $\text{canoe} \rightarrow \text{early}$     | Premise #4.             |
| 8. early                                       | Modus ponens on 6,7.    |

## 5) Rules of Inference for Quantifiers

- Based on the principle of *universal instantiation* – If some property is true of everything in a set, then it is true of any particular thing in the set.
  - e.g. For all real number  $x$ ,  $x^2 \geq 0$ .
- Two forms:

### 1. Universal Modus Ponens

|   |   |   |
|---|---|---|
| For all $x$ , if $P(x)$ , then $Q(x)$ .<br>$P(a)$ for a particular $a$ .<br>Therefore, $Q(a)$ . | <u>Example 1:</u><br>All humans are mortal.<br>Socrates is human.<br>Therefore, Socrates is mortal. | <u>Example 2:</u><br>For all even integer $x$ , its square is even.<br>A particular $k$ is even.<br>Therefore, $k^2$ is even. |
|---|---|---|

### 2. Universal Modus Tollens

|   |   |
|---|---|
| For all $x$ , if $P(x)$ , then $Q(x)$ .<br>$\sim Q(a)$ for a particular $a$ .<br>Therefore, $\sim P(a)$ . | <u>Example:</u><br>All humans are mortal.<br>Zeus is not mortal.<br>Therefore, Zeus is not human. |
|---|---|

- Validity of arguments with quantified statements.**

To say that an argument form is **valid** means the following: No matter what particular predicates are substituted for the predicate symbols in its premises, if the resulting premise statements are all true, then the conclusion is also true. An argument is called **valid** if, and only if, its form is valid.

HOWEVER, be careful in using either form in a proof – Both forms are **ONLY proving the instantiated value (a)**.

- Exercise: [Section 3.4, Exercise #19, p. 143]  
Rewrite the statement “No good cars are cheap” in the form “ $\forall x$ , if  $P(x)$  then  $\sim Q(x)$ .” Indicate whether each of the following arguments is valid or invalid, and justify your answers.
  - No good car is cheap.  
A Rimbaud is a good car.  
 $\therefore$  A Rimbaud is not cheap.
  - No good car is cheap.  
A Simbaru is not cheap.  
 $\therefore$  A Simbaru is a good car.

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## 2. Methods of Proof (§4.1-§4.7)

After an argument is determined to be valid, we can construct a proof. But before we proceed to proofs, we must remember that

- Validity of an argument is only about the form (e.g. Modus Ponens), and NOT about the content.  
e.g.  
If  $2 = 3$ , then I ate my hat.  
I ate my hat.  
 $\therefore 2 = 3$ .
- In constructing a proof, we apply reasoning based on the truth/falsity of the **domain** of the argument.

### 1) Some terminology

- **Axiom** -- a logical sentence/statement.
- **Theorem** -- a proposition that has been proved to be true. e.g. "For all real number  $x$ ,  $x^2 \geq 0$ ".
- **Lemma** -- a theorem that is not too interesting by itself but useful in proving other theorems. e.g. "For all real number  $x$ ,  $x^2 \geq x$ ".
- **Corollary** -- a theorem which follows from another theorem quickly. e.g. "For all real number  $x$ ,  $x^2 + 1 \geq 0$ ".

### 2) Various proof techniques

Example:

To prove a theorem "If  $P(x_1, \dots, x_n)$ , then  $Q(x_1, \dots, x_n)$ ", where  $x_1 \in D_1, \dots, x_n \in D_n$ .

- a. **Direct proof** and counter example
- Show the theorem is true for **all** values of  $x_1 \in D_1, \dots, x_n \in D_n$ , or show a counterexample  $(x_1, \dots, x_n)$  in which the theorem is false.
  - *Example:*

**Proposition:** If  $n$  is any even integer, then  $(-1)^n = 1$ .

Proof: Suppose  $n$  is any even integer. Then  $n = 2k$  for some integer  $k$ . Hence, by laws of exponents of algebra,

$$(-1)^n = (-1)^{2k} = ((-1)^2)^k = 1^k = 1. \text{ Therefore, the proposition is true.}$$

- b. **Proof by contradiction**

- To prove the theorem (to be true), show that assuming the **negation** of the theorem leads to a contradiction.
- Steps:
  1. Start with  $\neg(P(x_1, \dots, x_n) \rightarrow Q(x_1, \dots, x_n))$ , which is equivalent to  $P(x_1, \dots, x_n) \wedge \neg Q(x_1, \dots, x_n)$ .
  2. Then show there are no  $x_1, \dots, x_n$  which make that negated theorem true.
- *Example1:*

**Proposition:** For all integers  $n$ , if  $n^2$  is even, then  $n$  is even.

Proof: Suppose not. That is, [Negation of the theorem] suppose there exists an integer  $n$  such that  $n^2$  is even and  $n$  is odd. Since  $n$  is odd,  $n = 2k + 1$  for some integer  $k$ . Then,

$$n^2 = (2k + 1)^2 = (2k + 1)(2k + 1) = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$$

by the laws of algebra. Let  $m = 2k^2 + 2k$ . Then,  $m$  is an integer because 2 and  $k$  are integers, and sums and products of integers are integers. It follows by substitution that  $n^2 = 2m + 1$  for an integer  $m$ , and thus  $n^2$  is odd. Thus contradiction. [Hence, the supposition is false and the theorem is true.]

- *Example2:*

**Proposition:** Prove that  $2^{1/2}$  is irrational.

Proof: Suppose not. That is, [Negation of the theorem] suppose  $2^{1/2}$  is rational. Then  $2^{1/2} = p/q$  for some integer  $p$  and  $q$  with  $q \neq 0$  and  $\gcd(p, q) = 1$ .

Since  $2^{1/2} = p/q$ , we know  $(p/q)^2 = 2$  (i.e., square root of 2, to the power of 2). Then we get  $(p/q)^2 = p^2/q^2 = 2$ , and  $p^2 = 2 \cdot q^2$ . So  $p^2$  is even.

Let  $p = 2k$  for some integer  $k$ . So  $p^2 = (2k)^2 = 4k^2$ , which is equal to  $2 \cdot q^2$ . Dividing both sides by 2, we get  $2k^2 = q^2$ . Since  $k^2$  is an integer,  $q^2$  is even, therefore  $q$  is even. But then  $p$  and  $q$  have a common divisor, namely 2, so we have a contradiction.

#### c. Proof by contraposition

- Prove the theorem If  $P(x_1, \dots, x_n)$ , then  $Q(x_1, \dots, x_n)$  directly by showing (to be true), proving its contrapositive, If  $\sim Q(x_1, \dots, x_n)$ , then  $\sim P(x_1, \dots, x_n)$ .
- *Example:*

**Proposition:** For all integers  $n$ , if  $n^2$  is even, then  $n$  is even.

Proof: We show if  $n$  is odd,  $n^2$  is odd. Suppose  $n$  is any odd integer. By definition of odd,  $n = 2k + 1$  for some integer  $k$ . By substitution and algebra,

$$n^2 = (2k + 1)^2 = (2k + 1)(2k + 1) = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1.$$

Let  $m = 2k^2 + 2k$ . Then,  $m$  is an integer because 2 and  $k$  are integers, and sums and products of integers are integers. It follows by substitution that  $n^2 = 2m + 1$  for an integer  $m$ , and thus  $n^2$  is odd.

#### d. (Dis)proof by counterexample

- Prove a universally quantified theorem is false by giving a counterexample.
- *Example:*

**Proposition:** Every positive integer is the sum of the squares of two integers.

Proof: The theorem is false. For the positive integer 3, there is no way 3 can be expressed as the sum of the squares of two integers.

$3 = x^2 + y^2 \Rightarrow$  then both  $x^2 < 3$  and  $y^2 < 3 \Rightarrow x = 0$  or  $1$  and  $y = 0$  or  $1$ . But for either case,  $x^2 + y^2 < 3$ .

### 4) Exercises

- Give a proof by contradiction of the following statement:

For all real numbers  $x$  and  $y$ , if  $x + y \geq 2$ , then either  $x \geq 1$  or  $y \geq 1$ .

- b. [Section 4.6, Exercise #18, p. 206]

Fill in the blanks in the following proof by contraposition that for all integers  $n$ , if 5 does not divide  $n^2$  then 5 does not divide  $n$ .

Proof (by contraposition): [The contrapositive is: For all integers  $n$ , if  $5 \mid n$  then  $5 \mid n^2$ .]

Suppose  $n$  is any integer such that (a). [We must show that (b).] By definition of divisibility,  $n = \underline{(c)}$  for some integer  $k$ . By substitution,  $n^2 = \underline{(d)} = 5(5k^2)$ . But  $5k^2$  is an integer because it is a product of integers. Hence  $n^2 = 5 \cdot (\text{an integer})$ , and so (e) [as was to be shown].

- c. [Section 4.1, Exercise #47, p. 162] Determine whether the statement is true or false. Justify your answer with a proof or a counterexample.

If a sum of two integers is even, then one of the summands is even.