1.1 Propositions (Propositional Logic)

- A proposition is a statement that can be either true (T) or false (F), (but not both).

**Examples:**

- "The earth is flat." -- F
- "March has 31 days." -- T
- "Time flies like fruit flies." -- Not a proposition (it’s a metaphor)
- "Take CSC 400." -- Not a proposition (it’s a command)

- **Notation:** Lower case letters are often used to represent propositions.

**Examples:**

- $p$: "The earth is flat."
- $q$: "March has 31 days."

**Connectives (or operators)**

- Connectives are symbols that combine propositions. Propositions separated by connectives make a compound proposition.
- Basic connectives:
  1. "p and q" is the conjunction, noted "$p \land q$".
     e.g. "The earth is flat and March has 31 days."
  2. "p or q" is the disjunction, noted "$p \lor q$".
     e.g. "The earth is flat or March has 31 days."

  **NOTE:** The meaning of or here is inclusive, that is, if one is true, the truth of the other can be either true or false (i.e., not necessarily false). For example, "I will buy a car, or I will take a vacation."

  3. "not p" is the negation, noted "$\neg p" (or "\neg p" or "¬p").
     e.g. "The earth is not flat." or "It is not the case where the earth is flat."

- **Precedence and associativity of connectives**
  - **not** has the highest precedence, then **and**, then **or**.
  - **and** and **or** associate to the left -- group two propositions from the left
  - Parentheses are sometimes placed to force the order.

**Examples:**

- $p \land \neg q \land r$ means $(p \land (\neg q)) \land r$
- $p \lor \neg q \land r$ means $p \lor ((\neg q) \land r)$
- $\neg p \land \neg q$ means $(\neg p) \land (\neg q)$
\( \overline{p \land q} \) means \( \neg(p \land q) \)

**Truth Values & Truth Tables**

- **Truth values** of connectives (and, or, not)

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<th></th>
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<th>( p \land q )</th>
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<th>( p \lor q )</th>
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- **Exclusive-Or (\( \oplus \))**
  - The regular Or (\( \lor \)) is inclusive – it is true if either literal is true, or BOTH literals are true.
  - Another, more strict Or, is Exclusive-Or, denoted \( \oplus \). It is true strictly when EITHER literal is true, not both. The truth table is:

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<th>( p \oplus q )</th>
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- **Truth value of a compound proposition**

*Examples*: Suppose \( p, r \) are true and \( q \) is false. Evaluate the following propositions.

- \( \neg(p \land q) \)
- \( (\neg p) \land (\neg q) \)
- \( p \lor \neg q \land r \)

- **Truth table** lists truth values for ALL possible assignments of true/false

*Examples:*

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<th>( \neg(p \land q) )</th>
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### Tautologies and Contradictions
- A tautology is a statement that is always true regardless of the truth values of the individual statements.
- A contradiction a statement that is always false regardless of the truth values of the individual statements.

Example: \( p \lor \sim p \) (tautology) and \( p \land \sim p \) (contradiction)

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<tr>
<th>( p )</th>
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### 1.2 Conditional Propositions and Logical Equivalence

#### Conditional Propositions
- Conditional operator if:

  "if \( p \) then \( q \)" is the conditional proposition, noted "\( p \rightarrow q \)."

  - \( p \) is called the hypothesis or antecedent.
  - \( q \) is called the conclusion or consequent.

Example:
- \( p \): "I am rich."
- \( q \): "I buy a car."
- \( p \rightarrow q \): "If I were rich, then I buy a car." (or “If I am rich, I would buy a car.”)
• Truth table for $\rightarrow$

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By definition

• **Precedence** and associativity of if
  - $\neg$, then $\land$, then $\lor$, then $\rightarrow$
  - $\rightarrow$ associates to the left

**Examples:** Parenthesize the following statements

a) $p \lor \neg q \rightarrow r$

b) $p \rightarrow q \land \neg p \rightarrow r$

• **Necessary** and **sufficient conditions**
  - The conclusion expresses a necessary condition.
  - The hypothesis expresses a sufficient condition.

• Compound statements involving $\rightarrow$

**Examples:**

a. Suppose $p$, $r$ are true and $q$ is false. Evaluate the following propositions.

$\neg(p \rightarrow q)$

$(\neg p) \rightarrow (\neg q)$

$(p \land q) \rightarrow r$

• Truth table for $(p \rightarrow q) \rightarrow r$

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<th>$(p \rightarrow q) \rightarrow r$</th>
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• **Converse, Inverse and Contrapositive** of a conditional statement

For a conditional statement \( p \rightarrow q \),

1. converse is \( q \rightarrow p \).
2. inverse is \((\neg p) \rightarrow (\neg q)\).
3. contrapositive is \((\neg q) \rightarrow (\neg p)\).

**Some properties:**

a. Converse \((q \rightarrow p)\) is not logically equivalent to the original conditional statement \( p \rightarrow q \).

b. Inverse \((q \rightarrow p)\) is not logically equivalent to the original conditional statement \( p \rightarrow q \).

c. Contrapositive \(((\neg q) \rightarrow (\neg p))\) is **logically equivalent** to \( p \rightarrow q \).

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<th>inverse ((\neg p) \rightarrow (\neg q))</th>
<th>contrapositive ((\neg q) \rightarrow (\neg p))</th>
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• **Biconditional propositions -- "If and only if"**

  o When both \( p \rightarrow q \) and \( q \rightarrow p \) (converse) are true, it is said that "\( p \text{ if and only if } q \)";
    denoted \( p \leftrightarrow q \).

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**Logical Equivalence**

• Two statements \( P \) and \( Q \) are logically equivalent, denoted \( P \equiv Q \), when truth values in ALL rows in the truth tables are the same.

**Example:** \( p \rightarrow q \equiv \neg p \lor q \) --- SUPER IMPORTANT!!!
DeMorgan's Laws

- Logical equivalence for $p \land q \equiv \overline{p} \lor \overline{q}$, $p \lor q \equiv \overline{p} \land \overline{q}$.

Proof by truth table (for $-(p \land q)$ and $-p \lor -q$)

<table>
<thead>
<tr>
<th>$p$</th>
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<th>$-(p \land q)$</th>
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1.3 Quantifiers (Predicate Logic)

Propositional Function

- Propositions are NOT flexible -- no 'variables' in a statement. For example,
  - $p1$: "January has 31 days." -- true
  - $p2$: "February has 31 days." -- false
  - $p3$: "March has 31 days." -- true
  - $p4$, etc.

- A propositional function is a (logic) statement with variables.

**Definition:** Let $P(x)$ be a statement involving a variable $x$, and let $D$ be the set of values for $x$. If for each $x \in D$, $P(x)$ is a proposition then $P(x)$ is a propositional function with respect to $D$. $D$ is called the domain of discourse.

**Example:** $P(x)$: "The $x^{th}$ month of a year has 31 days", where $x$ is an integer $1 \leq x \leq 12$.

- The variable $x$ in a propositional function $P(x)$ is called a free variable.

**NOTE** (to be revised later):
- A propositional function is NOT a proposition -- it does not have T/F value by itself.
- T/F is obtained only after we plug in specific value of $x$.

**Example above:**
- $P(1)$ -- true
- $P(2)$ -- false, etc.

Quantified Statements

- Some propositional functions are quantified statements.
  - "For every $x$, $P(x)$" -- universally quantified
  - "For some $x$, $P(x)$" -- existentially quantified
1. Universally Quantified Statements ($\forall$)

- **Definition:** Let $P(x)$ be a propositional function over $D$. The statement "for every $x$ in $D$, $P(x)$" is said to be a **universally quantified statement**, noted $\forall x \in D, P(x)$.

- Different wording of statements
  - "For every $x$, $P(x)$"
  - "For each $x$, $P(x)$"
  - "For all $x$, $P(x)$"

*Example:* $P(x)$: "For every real number $x$, $x^2 \geq 0$"

- **Truth value** of $\forall x \in D, P(x)$
  - true -- if for every $x$ in $D$, $P(x)$ is true.
  - false -- if there is at least one $x$ in $D$ for which $P(x)$ is false -- a **counterexample**.

2. Existentially Quantified Statements ($\exists$)

- **Definition:** Let $P(x)$ be a propositional function over $D$. The statement "for some $x$ in $D$, $P(x)$" is said to be an **existentially quantified statement**, noted $\exists x \in D, P(x)$.

- Different wording of statements
  - "For some $x$, $P(x)$"
  - "For at least one $x$, $P(x)$"
  - "There exists $x$ such that $P(x)$"

*Example:* "There exists a real number $x$ such that $x^2 = 2$.

- Truth value of $\exists x \in D, P(x)$
  - true -- if there is at least one $x$ in $D$ for which $P(x)$ is true.
  - false -- if for all $x$ in $D$, $P(x)$ is false.

3. Quantified Statements as Propositions

- The variable $x$ in a quantified statement, $\forall x. P(x)$ or $\exists x. P(x)$, is called a **bound variable**.
- A quantified statement has a truth value -- although it is a propositional function.

**NOTE (revised):**

- A propositional function with **FREE** variables is NOT a proposition.
- A propositional function with **BOUND** variables (i.e., quantified statements) is a proposition.

4. Generalized DeMorgan's Laws

- Logical equivalency for negated quantified statements
  - $\forall x. \neg P(x) \iff \exists x. \neg P(x)$
    - $\neg(\forall x. P(x))$ --- "It is not the case that, for all $x$, $P(x)$ is true"
    - $\exists x. (\neg P(x))$ --- "There exists $x$ for which $P(x)$ is false."
5. Proving Quantified Statements

1. Proving a universally quantified statement “∀x. P(x)”
   - True -- by showing P(x) is true for ALL x.
     - IMPORTANT NOTE: You can NOT just plug in a few values of x and conclude the statement is true. You must pick a generic particular (but arbitrary chosen) value (x) and generalize.
   - False -- by showing a counterexample x in D for which P(x) is false (i.e., ∃x.(¬P(x)))

Examples: Prove or disprove:

   a. The sum of any two even integers is even.
      i. Proof: Suppose m and n are even integers. We must show that m + n is even. By definition of even, m = 2*r and n = 2*s for some integers r and s. Then,
         ii. m + n = 2*r + 2*s   ... by substitution
             = 2(r + s)      ... by factoring
         iii. Let k = r + s. Then, k is an integer because it is a sum of integers. Hence,
             iv. m + n = 2*k, where k is an integer. It follows by definition of even that m + n is even.
   b. For all real number x, x^2 - 1 > 0.
      i. Proof: The statement is false. A counterexample is x = 0. Here, 0 is a real number, but 0^2 - 1 = -1 <= 0 [NOTE: the negation of x^2 - 1 > 0 is x^2 - 1 <= 0].
   c. For all real number x, if x > 1, then x^2 - 1 > 0.
      \[ ∀x.(x ≠ 0) → \frac{1}{x^2} > \frac{1}{x^3} \]
   d.

2. Proving an existentially quantified statement “∃x. P(x)”
   - True -- by showing there exists at least one x in D such that P(x) is true.
   - False -- by showing for all x, P(x) is false (i.e., ∀x.(¬P(x)))

Examples: Prove or disprove

   a. For some real number x, x > 5 and x < 10
   b. For some real number x, x > 5 and x < 4
Multiple Quantifiers and Variables

- Statement with two quantifiers and variables
  - $\forall x \forall y. P(x, y)$
  - $\exists x \exists y. P(x, y)$
  - $\forall x \exists y. P(x, y)$
  - $\exists x \forall y. P(x, y)$

- Negations
  - Derive equivalent forms by applying DeMorgan's law several times.

  e.g. $\overline{\forall x \forall y. P(x, y)} \equiv \exists x \exists y. P(x, y) \equiv \exists x \exists y. \overline{P(x, y)}$

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<th>Negation</th>
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<tbody>
<tr>
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<td>$\forall x \forall y. \overline{P(x, y)}$</td>
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<td>$\exists x \forall y. P(x, y)$</td>
<td>$\forall x \exists y. \overline{P(x, y)}$</td>
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- Examples: Prove or disprove:
  - a. $\forall x, \forall y. x^2 + 2y > 4$
  - b. $\exists x, \exists y. x^2 + 2y > 4$
  - c. $\forall x, \exists y. x^2 + 2y > 4$
  - d. $\exists x, \forall y. x^2 + 2y > 4$
  - e. $\forall x, \forall y. \text{if } x < y, \text{ then } x^2 + 2y > 4$
  - f. $\exists x, \exists y. \text{if } x < y, \text{ then } x^2 + 2y > 4$
  - g. $\forall x, \exists y. \text{if } x < y, \text{ then } x^2 + 2y > 4$
  - h. $\exists x, \forall y. \text{if } x < y, \text{ then } x^2 + 2y > 4$