400 lecture note #1

[Ch 2, 3] Logic and Proofs

1.1 Propositions (Propositional Logic)

- A **proposition** is a statement that can be either true (T) or false (F), (but not both).

  Examples:

  - "The earth is flat." -- F
  - "March has 31 days." -- T
  - "Time flies like fruit flies." -- Not a proposition (it's a metaphor)
  - "Take CSC 400." -- Not a proposition (it's a command)
- **Notation:** Lower case letters are often used to represent propositions.

  Examples:

  - p: "The earth is flat."
  - q: "March has 31 days."

Connectives (or operators)

- Connectives are symbols that combine propositions. Propositions separated by connectives make a **compound** proposition.
- Basic connectives:
  1. "p **and** q" is the conjunction, noted "$p \wedge q$".
     e.g. "The earth is flat and March has 31 days."
  2. "p **or** q" is the disjunction, noted "$p \vee q$".
     e.g. "The earth is flat or March has 31 days."

     NOTE: The meaning of or here is **inclusive**, that is, if one is true, the truth of the other can be either true or false (i.e., not necessarily false). For example, "I will buy a car, or I will take a vacation."
  3. "**not** p" is the negation, noted " $\overline{p}$ " (or "$\neg p$" or "$\sim p$").
     e.g. "The earth is not flat." or "It is not the case where the earth is flat."
- **Precedence** and associativity of connectives
  - **not** has the highest precedence, then **and**, then **or**.
  - **and** and **or** associate to the left -- group two propositions from the left
  - Parentheses are sometimes placed to force the order.

  Examples:

  - $p \wedge \neg q \wedge r$ means $(p \wedge (\neg q)) \wedge r$
  - $p \vee \neg q \wedge r$ means $p \vee ((\neg q) \wedge r)$
  - $\overline{p} \wedge \overline{q}$ means $(\neg p) \wedge (\neg q)$

- $\overline{p \wedge q}$ means $\neg(p \wedge q)$

Truth Values & Truth Tables

- **Truth values** of connectives (and, or, not)

$p \wedge q$

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

$p \vee q$

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

$\neg p$

p	$\neg p$
T	F
F	T

- **Exclusive-Or** ($\oplus$)
  - The regular Or ($\vee$) is inclusive – it is true if either literal is true, or BOTH literals are true.
  - Another, more strict Or, is Exclusive-Or, denoted $\oplus$. It is true strictly when EITHER literal is true, not both. The truth table is:

$p \oplus q$

p	q	$p \oplus q$
T	T	**F**
T	F	T
F	T	T
F	F	F

- Truth value of a compound proposition

Examples: Suppose p, r are true and q is false. Evaluate the following propositions.

$\neg(p \wedge q)$	
$(\neg p) \wedge (\neg q)$	
$p \vee \neg q \wedge r$	

- **Truth table** lists truth values for ALL possible assignments of true/false

Examples:

p	q	$\neg(p \wedge q)$
T	T	
T	F	
F	T	
F	F	

p	q	$(\neg p) \wedge (\neg q)$
T	T	
T	F	
F	T	
F	F	

p	q	r	$p \vee \neg q \wedge r$
T	T	T	
T	T	F	
T	F	T	
T	F	F	
F	T	T	
F	T	F	
F	F	T	
F	F	F	

- **Tautologies** and **Contradictions**
  - A tautology is a statement that is always **true** regardless of the truth values of the individual statements.
  - A contradiction a statement that is always **false** regardless of the truth values of the individual statements.

  Example: p or $\sim p$ (tautology) and p and $\sim p$ (contradiction)

p	$p \vee \sim p$	$p \wedge \sim p$
T	T	F
F	T	F

1.2 Conditional Propositions and Logical Equivalence

Conditional Propositions

- Conditional operator **if**:

  "**if** p **then** q" is the **conditional** proposition, noted "$p \rightarrow q$".

  - p is called the **hypothesis** or antecedent.
  - q is called the **conclusion** or consequent.

  Example:
  - p : "I am rich."
  - q : "I buy a car."
  - $p \rightarrow q$: "If I were rich, then I buy a car." (or "If I am rich, I would buy a car.")

- Truth table for →

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

}By definition

- **Precedence** and associativity of if
    - ¬, then ∧, then ∨, **then** →
    - → associates to the left

Examples: Parenthesize the following statements

a) $p \vee \neg q \to r$	
b) $p \to q \wedge \neg p \to r$	

- **Necessary** and **sufficient conditions**
    - The conclusion expresses a <u>necessary</u> condition.
    - The hypothesis expresses a <u>sufficient</u> condition.

- Compound statements involving →

Examples:

a. Suppose p, r are true and q is false. Evaluate the following propositions.

$\neg(p \to q)$	
$(\neg p) \to (\neg q)$	
$(p \wedge q) \to r$	

- Truth table for $(p \to q) \to r$

p	q	r	$(p \to q) \to r$
T	T	T	
T	T	F	
T	F	T	
T	F	F	
F	T	T	
F	T	F	
F	F	T	
F	F	F	

- **Converse**, **Inverse** and **Contrapositive** of a conditional statement

  For a conditional statement $p \to q$,

  1. converse is $\boldsymbol{q \to p}$.
  2. inverse is $\boldsymbol{(\neg p) \to (\neg q)}$.
  3. contrapositive is $\boldsymbol{(\neg q) \to (\neg p)}$.

  Some properties:

  a. Converse ($q \to p$) is not logically equivalent to the original conditional statement $p \to q$.
  b. Inverse ($q \to p$) is not logically equivalent to the original conditional statement $p \to q$.
  c. Contrapositive ($((\neg q) \to (\neg p))$) is **logically equivalent** to $p \to q$.

p	q	$p \to q$	converse $q \to p$	inverse $\boldsymbol{(\neg p) \to (\neg q)}$.	contrapositive $\boldsymbol{(\neg q) \to (\neg p)}$
T	T	T	T	T	T
T	F	F	T	T	F
F	T	T	F	F	T
F	F	T	T	T	T

- **Biconditional propositions -- "If and only if"**
  - When both $p \to q$ and $q \to p$ (converse) are true, it is said that "p **if and only if** q", denoted $p \leftrightarrow q$.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Logical Equivalence

- Two statements P and Q are logically equivalent, denoted P ≡ Q, when truth values in ALL rows in the truth tables are the same.

  Example: $\boldsymbol{p \to q \equiv \sim p \vee q}$ --- SUPER IMPORTANT!!!

p	q	$p \to q$	$\neg p \vee q$
T	T		
T	F		
F	T		
F	F		

DeMorgan's Laws

- Logical equivalence for $\overline{p \wedge q} \equiv \overline{p} \vee \overline{q}$, $\overline{p \vee q} \equiv \overline{p} \wedge \overline{q}$.

  Proof by truth table (for $\neg(p \wedge q)$ and $\neg p \vee \neg q$)

p	q	$\neg(p \wedge q)$	$\neg p \vee \neg q$
T	T	F	
T	F	T	
F	T	T	
F	F	T	

1.3 Quantifiers (Predicate Logic)

Propositional Function

- Propositions are NOT flexible -- no 'variables' in a statement. For example,
  - *p1*: "January has 31 days." -- true
  - *p2*: "February has 31 days." -- false
  - *p3*: "March has 31 days." -- true
  - *p4*..., etc.
- A **propositional function** is a (logic) statement with variables.

  Definition: Let P(x) be a statement involving a variable x, and let D be the set of values for x. If for each x ∈ D, P(x) is a proposition then P(x) is a propositional function with respect to D. D is called the domain of discourse.

  Example: P(x): "The x^{th} month of a year has 31 days", where x is an integer 1 <= x <= 12.

- The variable *x* in a propositional function P(x) is called a **free variable**.
- **NOTE** (to be revised later):
  - A propositional function is NOT a proposition -- it does not have T/F value **by itself**.
  - T/F is obtained only after we plug in specific value of x.

  Example above:
  - P(1) -- true
  - P(2) -- false, etc.

Quantified Statements

- Some propositional functions are quantified statements.
  - "For every x, P(x)" -- universally quantified
  - "For some x, P(x)" -- existentially quantified

1. Universally Quantified Statements (∀)

- **Definition:** Let P(x) be a propositional function over D. The statement "for every x in D, P(x)" is said to be a universally quantified statement, noted $\forall x \in D, P(x)$.
- Different wording of statements
  - "For every x, P(x)"
  - "For each x, P(x)"
  - "For all x, P(x)"

  Example: P(x): "For every real number x, x^2 >= 0"

- **Truth value** of $\forall x \in D, P(x)$
  - **true** -- if for every x in D, P(x) is true.
  - **false** -- if there is at least one x in D for which P(x) is false -- a **counterexample.**

2. Existentially Quantified Statements (∃)

- **Definition:** Let P(x) be a propositional function over D. The statement "for some x in D, P(x)" is said to be an existentially quantified statement, noted $\exists x \in D, P(x)$.
- Different wording of statements
  - "For some x, P(x)"
  - "For at least one x, P(x)"
  - "There exists x such that P(x)"

  Example: "There exists a real number x such that $x^2 = 2$."

- Truth value of $\exists x \in D, P(x)$
  - **true** -- if there is at least one x in D for which P(x) is true.
  - **false** -- if for all x in D, P(x) is false.

3. Quantified Statements as Propositions

- The variable *x* in a quantified statement, ∀x.P(x) or ∃x.P(x), is called a **bound variable**.
- A quantified statement has a truth value -- although it is a propositional function.

  NOTE (revised):

  - A propositional function with FREE variables is NOT a proposition.
  - A propositional function with BOUND variables (i.e., quantified statements) is a proposition.

4. Generalized DeMorgan's Laws

- Logical equivalency for negated quantified statements
  - $\overline{\forall x.P(x)} \Leftrightarrow \exists x.\overline{P(x)}$
    - ¬(∀x.P(x)) –– "It is not the case that, for all x, P(x) is true"
    - ∃x.(¬P(x)) –– "There exists x for which P(x) is false."

- $\overline{\exists x.P(x)} \Leftrightarrow \forall x.\overline{P(x)}$
  - ¬(∃x.P(x)) –– "It is not the case that there exists x for which P(x) is true."
  - ∀x.(¬P(x)) –– "For all x, P(x) is false."

5. Proving Quantified Statements

1. Proving a **universally quantified statement "∀x. P(x)"**
   - True -- by showing P(x) is true for **ALL** x.
     - IMPORTANT NOTE:
       You can NOT just plug in a few values of x and conclude the statement is true. You must pick a **generic particular (but arbitrary chosen)** value (x) and generalize.
   - False -- by showing a **counterexample** x in D for which P(x) is false (i.e., ∃x.(¬P(x)))

   Examples: Prove or disprove:

   a. The sum of any two even integers is even.
      i. *Proof:* Suppose m and n are even integers. We must show that m + n is even. By definition of even, m = 2*r and n = 2*s for some integers r and s. Then,
      ii. m + n = 2*r + 2*s ... by substitution
          = 2(r + s) ... by factoring
      iii. Let k = r + s. Then, k is an integer because it is a sum of integers. Hence,
      iv. m + n = 2*k, where k is an integer. It follows by definition of even that m + n is even.
   b. For all real number x, $x^2 - 1 > 0$.
      i. *Proof:* The statement is false. A counterexample is x = 0. Here, 0 is a real number, but $0^2 - 1 = -1 <= 0$ [NOTE: the **negation** of $x^2 - 1 > 0$ is $x^2 - 1$ **<=** 0].

   c. For all real number x, if x > 1, then $x^2 - 1 > 0$.
   d. $\forall x.(x \neq 0) \rightarrow \frac{1}{x^2} > \frac{1}{x^3}$

2. Proving an **existentially quantified statement "∃x. P(x)"**
   - True -- by showing there exists at least one x in D such that P(x) is true.
   - False -- by showing for all x, P(x) is false (i.e., ∀x.(¬P(x)))

   Examples: Prove or disprove

   a. For some real number x, x > 5 and x < 10

   b. For some real number x, x > 5 and x < 4

Multiple Quantifiers and Variables

- Statement with two quantifiers and variables
  - $\forall x.\forall y.P(x,y)$
  - $\exists x.\exists y.P(x,y)$
  - $\forall x.\exists y.P(x,y)$
  - $\exists x.\forall y.P(x,y)$
- Negations
  - Derive equivalent forms by applying DeMorgan's law several times.

    e.g. $\overline{\forall x \forall y.P(x,y)} \equiv \exists x \overline{\forall y.P(x,y)} \equiv \exists x \exists y \overline{P(x,y)}$

	Negation
$\forall x.\forall y.P(x,y)$	$\exists x.\exists y.\overline{P(x,y)}$
$\exists x.\exists y.P(x,y)$	$\forall x.\forall y.\overline{P(x,y)}$
$\forall x.\exists y.P(x,y)$	$\exists x.\forall y.\overline{P(x,y)}$
$\exists x.\forall y.P(x,y)$	$\forall x.\exists y.\overline{P(x,y)}$

- *Examples:* Prove or disprove:
  a. $\forall x.\forall y.\ x^2 + 2y > 4$
  b. $\exists x.\exists y.\ x^2 + 2y > 4$
  c. $\forall x.\exists y.\ x^2 + 2y > 4$
  d. $\exists x.\forall y.\ x^2 + 2y > 4$
  e. $\forall x.\forall y.$ if $x < y$, then $x^2 + 2y > 4$
  f. $\exists x.\exists y.$ if $x < y$, then $x^2 + 2y > 4$
  g. $\forall x.\exists y.$ if $x < y$, then $x^2 + 2y > 4$
  h. $\exists x.\forall y.$ if $x < y$, then $x^2 + 2y > 4$