Lecture Note #8 (Recurrence Relations)
Exercises

===

(1) Iteration Method

<u>More Examples</u>:

a. T(n) = -3*T(n-1), T(0) = 2

<u>Solution</u>: By iteration.

$T(n) = (-3) \cdot T(n-1)$
$= (-3) \cdot [(-3) \cdot T(n-2)]$
$= (-3)^2 \cdot T(n-2)$
$= (-3)^2 \cdot [(-3) \cdot T(n-3)]$
$= (-3)^3 \cdot T(n-3)$
$= \cdots$
$= (-3)^k \cdot T(n-k)$
$= \cdots$
$= (-3)^n \cdot T(0)$ $\cdots$ because $n - k = 0$, which gives $k = n$
$= (-3)^n \cdot (2)$

So the closed form is $C(n) = (-3)^n \cdot (2)$ for all integer $n \geq 0$.

---

b. T(n) = T(n-1) + 1, T(0) = 0

<u>Solution</u>: By iteration.

$T(n) = T(n-1) + 1$
$= [T(n-1) + 1] + 1$
$= T(n-1) + 2$
$= \cdots$
$= T(n-k) + k$
$= \cdots$
$= T(0) + n$
$= n$

So the closed form is $C(n) = n$ for all integer $n \geq 0$.

---

c. T(n) = 2*T(n-1) + 3, T(0) = 0

<u>Solution</u>: By iteration.

$T(n) = 2 \cdot T(n-1) + 3$
$= 2 \cdot [2 \cdot T(n-2) + 3] + 3$
$= 2^2 \cdot T(n-2) + (2) \cdot 3 + 3$
$= 2^2 \cdot [2 \cdot T(n-3) + 3] + (2) \cdot 3 + 3$

$= 2^3 \cdot T(n-3) + 2^2 \cdot 3 + 2^1 \cdot 3 + 2^0 \cdot 3$
$= \cdots$
$= 2^k \cdot T(n-k) + \sum_{i=0}^{k-1} 3 \cdot 2^i$
$= 2^k \cdot T(n-k) + 3 \cdot \sum_{i=0}^{k-1} 2^i$
$= \cdots$
$= 2^n \cdot T(0) + 3 \cdot \sum_{i=0}^{n-1} 2^i$
$= 2^n \cdot T(0) + 3 \cdot \frac{2^n - 1}{2 - 1}$
$= 2^n \cdot 0 + 3 \cdot (2^n - 1)$
$= 3 \cdot (2^n - 1)$

So the closed form is $C(n) = 3 \cdot (2^n - 1)$ for all integer $n \geq 0$.

---

d. T(n) = T(n/2) + 1, T(1) = 0

Solution: By iteration.

$T(n) = T\left(\frac{n}{2}\right) + 1$
$= \left[T\left(\frac{n}{2^2}\right) + 1\right] + 1$
$= T\left(\frac{n}{2^2}\right) + 2$
$= \left[T\left(\frac{n}{2^3}\right) + 1\right] + 2$
$= T\left(\frac{n}{2^3}\right) + 3$
$= \cdots$
$= T\left(\frac{n}{2^k}\right) + k$
$= \cdots$
$= T(1) + \log_2 n$ ...because $2^k = n$, which gives $k = \log_2 n$
$= \log_2 n$

So the closed form is $C(n) = \log_2 n$ for all integers $n \geq 1$.

Proving the correctness of the derived formula

* **Example 2**: Show if T(1), T(2),.., T(n) is the sequence defined by

T(n) = -3*T(n-1) for all integers n >= 1, and T(0) = 2

Then the closed form is $T(n) = 2*(-3)^n$, for all n >= 0

Proof: By induction.

Basis step (n = 0):

- Recurrence: T(0) = 2, as given.
- Closed form: $T(0) = 2*(-3)^0 = 2*1 = 2$.

Therefore, the closed form holds for n = 0. … (A)

Inductive step:

Assume the closed form holds for n-1, that is, $T(n-1) = 2*(-3)^{n-1}$, for all integers n >= 1.
Show the closed form holds for n as well, that is, $T(n) = 2*(-3)^n$

$T(n) = (-3) \cdot T(n-1)$ ... by recurrence relation
$= (-3) \cdot [2 \cdot (-3)^{n-1}]$ ... by inductive hypothesis
$= 2 \cdot (-3)^n$ ... as to be shown ... (B)

By (A) and (B), we can conclude that the closed form holds for all integers n $>=$ 0. QED.