Some Fundamental Mathematical Concepts

1. Variables

- A variable is a place holder, which takes on some value.
- E.g. "*x*" in the statement "Is there a number *x* with the property that $2x + 3 = x^2$?"

2. Universal, conditional and existential statements

A **universal statement** says that a certain property is true for all elements in a set. (For example: *All positive numbers are greater than zero.*)

A **conditional statement** says that if one thing is true then some other thing also has to be true. (For example: *If 378 is divisible by 18, then 378 is divisible by 6.*)

Given a property that may or may not be true, an **existential statement** says that there is at least one thing for which the property is true. (For example: *There is a prime number that is even.*)

- Universal statement -- **"For all"** (or "For every"), and a symbol ∀

  e.g. "All humans are mortal."
  - "**For all** humans x, x is mortal." (using a variable)
  - "**∀x**. if x is a human, then x is mortal." (for-all and conditional)
- Existential statement – "**There exists**" (or "Some"), and a symbol ∃

  e.g. "Somebody in this room has a Japanese passport."
  - "**There exists** a person x in this room such that x has a Japanese passport." (using a variable)
  - "**∃x**. x is a person in this room and x has a Japanese passport." (there-exists and conjunction)
- Example 1.1.2 (p. 3): Fill in the blanks to rewrite the following statement

  "For all real numbers x, if x is nonzero then x^2 is positive."

  a. If a real number is nonzero, then its square ______.
  b. For all nonzero real numbers x, _____.
  c. If x _____, then _____.
  d. The square of any nonzero real number is _____.
  e. All nonzero real numbers have _____.

3. **Set notation**

• Notation

If S is a set, the notation $x \in S$ means that x is an element of S. The notation $x \notin S$ means that x is not an element of S. A set may be specified using the **set-roster notation** by writing all of its elements between braces. For example, **{1, 2, 3}** denotes the set whose elements are 1, 2, and 3. A variation of the notation is sometimes used to describe a very large set, as when we write **{1, 2, 3, . . . , 100}** to refer to the set of all integers from 1 to 100. A similar notation can also describe an infinite set, as when we write **{1, 2, 3, . . .}** to refer to the set of all positive integers. (The symbol **. . .** is called an **ellipsis** and is read "and so forth.")

- E.g. A = {1, 2, 3, 4}, B = {1, 4, 3, 2}, C = {3, 1, 2, 3, 4, 1}
- NOTE:
  - o Order of elements does not matter.
  - o Duplicates are ignored (i.e, only one instance is counted).
- Therefore, all sets above are same/equivalent … A = B = C.

4. **Set of numbers**

Symbol	Set
R	set of all real numbers
Z	set of all integers
Q	set of all rational numbers, or quotients of integers

- Some examples:
  - o R => 0.0, 1.38, 0.666666666666666..
  - o Z => 0, 183, -61
  - o Q => 1/3, -(2/5)

5. **'Set builder' notation**

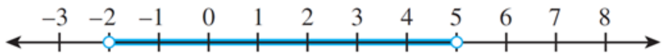
• Set-Builder Notation

Let S denote a set and let $P(x)$ be a property that elements of S may or may not satisfy. We may define a new set to be **the set of all elements x in S such that $P(x)$ is true**. We denote this set as follows:

$$\{x \in S \mid P(x)\}$$

the set of all ↗ ({) such that ↖ (|)

a. $\{x \in \mathbf{R} \mid -2 < x < 5\}$ is the open interval of real numbers (strictly) between –2 and 5. It is pictured as follows:



b. $\{x \in \mathbf{Z} \mid -2 < x < 5\}$ is the set of all integers (strictly) between –2 and 5. It is equal to the set $\{-1, 0, 1, 2, 3, 4\}$.

c. Since all the integers in $\mathbf{Z}^+$ are positive, $\{x \in \mathbf{Z}^+ \mid -2 < x < 5\} = \{1, 2, 3, 4\}$.

6. Subsets

• Definition

If A and B are sets, then A is called a **subset** of B, written $\boldsymbol{A \subseteq B}$, if, and only if, every element of A is also an element of B.

Symbolically:

$A \subseteq B$ means that For all elements x, if $x \in A$ then $x \in B$.

The phrases A *is contained in* B and B *contains* A are alternative ways of saying that A is a subset of B.

$A \not\subseteq B$ means that There is at least one element x such that $x \in A$ and $x \notin B$.

• Definition

Let A and B be sets. A is a **proper subset** of B if, and only if, every element of A is in B but there is at least one element of B that is not in A.

- Distinctions between subset, proper subset and membership relations:
  - $2 \in \{1,2,3,4\}$
  - $\{2\} \subseteq \{1,2,3,4\}$
  - $\{2\} \subset \{1,2,3,4\}$
  - $\{2\} \in \{1,\{2\},\{3,4\}\}$ => {2} is a singleton set, and it's a member in the larger set
  - $\{2,5\} \nsubseteq \{1,2,3,4\}$ => negation

7. Cartesian Products

• Notation

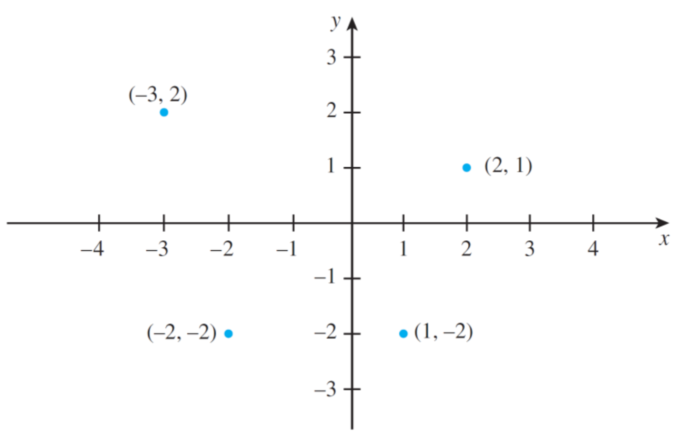
Given elements a and b, the symbol (a, b) denotes the **ordered pair** consisting of a and b together with the specification that a is the first element of the pair and b is the second element. Two ordered pairs (a, b) and (c, d) are equal if, and only if, $a = c$ and $b = d$. Symbolically:

$$(a, b) = (c, d) \quad \text{means that} \quad a = c \text{ and } b = d.$$

• Definition

Given sets A and B, the **Cartesian product of A and B**, denoted $\boldsymbol{A \times B}$ and read "A cross B," is the set of all ordered pairs (a, b), where a is in A and b is in B. Symbolically:

$$\boldsymbol{A \times B} = \{(a, b) \mid a \in A \text{ and } b \in B\}.$$



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Definitions in Elementary Number Theory

1. Even and Odd Integers

• Definitions

An integer n is **even** if, and only if, n equals twice some integer. An integer n is **odd** if, and only if, n equals twice some integer plus 1.

Symbolically, if n is an integer, then

$$n \text{ is even} \quad \Leftrightarrow \quad \exists \text{ an integer } k \text{ such that } n = 2k.$$

$$n \text{ is odd} \quad \Leftrightarrow \quad \exists \text{ an integer } k \text{ such that } n = 2k + 1.$$

2. **Prime numbers**

• Definition

An integer n is **prime** if, and only if, $n > 1$ and for all positive integers r and s, if $n = rs$, then either r or s equals n. An integer n is **composite** if, and only if, $n > 1$ and $n = rs$ for some integers r and s with $1 < r < n$ and $1 < s < n$.

In symbols:

n is prime $\Leftrightarrow$ $\forall$ positive integers r and s, if $n = rs$ then either $r = 1$ and $s = n$ or $r = n$ and $s = 1$.

n is composite $\Leftrightarrow$ $\exists$ positive integers r and s such that $n = rs$ and $1 < r < n$ and $1 < s < n$.

3. **Rational numbers**

• Definition

A real number r is **rational** if, and only if, it can be expressed as a quotient of two integers with a nonzero denominator. A real number that is not rational is **irrational.** More formally, if r is a real number, then

$$r \text{ is rational} \Leftrightarrow \exists \text{ integers } a \text{ and } b \text{ such that } r = \frac{a}{b} \text{ and } b \neq 0.$$

Theorem 4.2.1

Every integer is a rational number.

4. **Divisibility**

• Definition

If n and d are integers and $d \neq 0$ then

n is **divisible by** d if, and only if, n equals d times some integer.

Instead of "n is divisible by d," we can say that

n **is a multiple of** d, or
d **is a factor of** n, or
d **is a divisor of** n, or
d **divides** n.

The notation **d | n** is read "d divides n." Symbolically, if n and d are integers and $d \neq 0$:

$$d \mid n \quad \Leftrightarrow \quad \exists \text{ an integer } k \text{ such that } n = dk.$$

a. Is 21 divisible by 3? **b.** Does 5 divide 40? **c.** Does 7 | 42? **d.** Is 32 a multiple of −16? **e.** Is 6 a factor of 54? **f.** Is 7 a factor of −7?	**a.** Yes, 21 = 3 • 7. **b.** Yes, 40 = 5 • 8. **c.** Yes, 42 = 7 • 6. **d.** Yes, 32 = (−16) • (−2). **e.** Yes, 54 = 6 • 9. **f.** Yes, −7 = 7 • (−1).

5. Quotient-Remainder Theorem

Theorem 4.4.1 The Quotient-Remainder Theorem

Given any integer n and positive integer d, there exist unique integers q and r such that

$$n = dq + r \quad \text{and} \quad 0 \le r < d.$$

• Definition

Given an integer n and a positive integer d,

$$\textbf{\textit{n div d}} = \text{the integer quotient obtained when } n \text{ is divided by } d \text{, and}$$

$$\textbf{\textit{n mod d}} = \text{the nonnegative integer remainder obtained when } n \text{ is divided by } d.$$

Symbolically, if n and d are integers and $d > 0$, then

$$n \textit{ div } d = q \quad \text{and} \quad n \textit{ mod } d = r \quad \Leftrightarrow \quad n = dq + r$$

where q and r are integers and $0 \le r < d$.

6. Floor and Ceiling

• Definition

Given any real number x, the **floor of x,** denoted $\lfloor x \rfloor$, is defined as follows:

$$\lfloor x \rfloor = \text{that unique integer } n \text{ such that } n \le x < n + 1.$$

Symbolically, if x is a real number and n is an integer, then

$$\lfloor x \rfloor = n \quad \Leftrightarrow \quad n \le x < n + 1.$$

• Definition

Given any real number x, the **ceiling of x,** denoted $\lceil x \rceil$, is defined as follows:

$$\lceil x \rceil = \text{that unique integer } n \text{ such that } n - 1 < x \le n.$$

Symbolically, if x is a real number and n is an integer, then

$$\lceil x \rceil = n \quad \Leftrightarrow \quad n - 1 < x \le n.$$