Lecture Note #3 (Mathematical Induction)
Exercises

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2) Weak form of Mathematical Induction

#1. Show 1 + 3 + 5 + .. + (2n-1) = n^2.

<u>Proof</u>: The equation (to be proven) can be written using sequence notation as:

$$\sum_{i=1}^{n} 2i - 1 = n^2$$

We show by induction on n.

1) Basis step: When n = 1,

   LHS (Left-hand side) = $\sum_{i=1}^{1} 2i - 1 = 2 \cdot 1 - 1 = 1$
   RHS (Right-hand side) = 1^2 = 1.
   Therefore, we have LHS = RHS. … (A)

2) Inductive step:
   [Inductive hypothesis]
   Assume the equation holds for some integer k which is >= 1. That is, $\sum_{i=1}^{k} 2i - 1 = k^2$.
   [Inductive statement]
   We show that the equation holds for k+1, that is, $\sum_{i=1}^{k+1} 2i - 1 = (k+1)^2$.

   LHS
   $= \sum_{i=1}^{k+1} 2i - 1$
   $= \sum_{i=1}^{k} 2i - 1 + (2(k+1) - 1)$ … by definition of this sequence
   $= k^2 + (2(k+1) - 1)$ … by inductive hypothesis
   $= k^2 + 2k + 1$ … by algebra
   $= (k+1)^2$

   $RHS = (k+1)^2$ … as to be shown in inductive statement

   Thus we get LHS = RHS. … (B)

   By (A) and (B), we can conclude that the statement is true (for all integers n >= 1). QED.

#2. Show $1 + 2 + 2^2 + 2^3 + \ .. + 2^n = 2^{n+1}-1$.

Proof: The equation (to be proven) can be written using sequence notation as:

$$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$$

We show by induction on n.

1) Basis step: When n = 0,

   LHS = $\sum_{i=0}^{0} 2^i = 2^0 = 1$
   RHS = $2^{0+1} - 1 = 2^1 - 1 = 2 - 1 = 1$
   Therefore, we have LHS = RHS. … (A)

2) Inductive step:
   [Inductive hypothesis]
   Assume the equation holds for some integer k which is >= 0. That is, $\sum_{i=0}^{k} 2^i = 2^{k+1} - 1$.
   [Inductive statement]
   We show that the equation holds for k+1, that is, $\sum_{i=0}^{k+1} 2^i = 2^{k+2} - 1$.

$$\begin{aligned}
&LHS \\
&= \sum_{i=0}^{k+1} 2^i \\
&= \sum_{i=0}^{k} 2^i + 2^{k+1} \text{ ... by definition of this sequence} \\
&= (2^{k+1} - 1) + 2^{k+1} \text{ ... by inductive hypothesis} \\
&= 2 \cdot 2^{k+1} - 1 \\
&= 2^{k+2} - 1
\end{aligned}$$

$$RHS = 2^{k+2} - 1 \text{ .. as to be shown in inductive statement}$$

   Thus we get LHS = RHS. … (B)

   By (A) and (B), we can conclude that the statement is true (for all integers n >= 0). QED.

#3. Show n! >= 2^{n-1} for all integer n >= 1.

Proof: We show by induction on n.

1) Basis step: When n = 1,

LHS = n! = 1! = 1
RHS = $2^{n-1} = 2^{1-1} = 2^0 = 1$.
Therefore, we have LHS >= RHS. … (A)

2) Inductive step:
[Inductive hypothesis]
Assume the equation holds for some integer k which is >= 1. That is, k! >= 2^{k-1}.
[Inductive statement]
We show that the equation holds for k+1, that is, (k+1)! >= 2^k.

LHS
$= (k+1)!$
$= (k+1) \cdot k!$ … by definition of factorial
$\geq (k+1) \cdot 2^{k-1}$ … by inductive hypothesis

But since k >= 1 (by inductive assumption), we know that k+1 >= 2. Using that in the previous inequation, we get:

$(k+1) \cdot 2^{k-1} \geq 2 \cdot 2^{k-1} = 2^k$, which is the RHS of the inductive statement.
Therefore we have LHS >= RHS … (B)

By (A) and (B), we can conclude that the statement is true (for all integers n >= 1). QED.

#4. Show 5^n-1 is divisible by 4, for n=1,2,…

Proof: We show by induction on n.

1) Basis step: When n = 1,

   5^1-1 = 4, and 4 is divisible by 4. … (A)

2) Inductive step:
   [Inductive hypothesis] Assume 5^k-1 is divisible by 4.
   [Inductive statement] Show 5^{k+1}-1 is divisible by 4 as well.

   $$5^{k+1} - 1 = 5 \cdot 5^k - 1 = 5 \cdot (5^k - 1) + 4$$

   Here, since 5^k-1 is divisible by 4 by inductive hypothesis, it can be written as 4*a, where a is an integer. Substituting that in the previous expression,

   $$5 \cdot (5^k - 1) + 4 = 5 \cdot (4a) + 4 = 4 \cdot (5a + 1)$$

   Since 5a+1 is an integer, $4 \cdot (5a + 1)$ is divisible by 4. … (B)

   By (A) and (B), we can conclude that the statement is true (for all integers n >= 1). QED.

#5. Show that, for all n >= 4 (where n is also an integer), n cents can be obtained using 2-cent and 5-cent coins (only).

Proof: We show by induction on n.

1) Basis step: When n = 4, we can make 4 cents by two 2-cent coins. … (A)

2) Inductive step:
   [Inductive hypothesis] Assume k cents can be obtained using 2-cent and 5-cent coins (only), k >= 5.
   [Inductive statement] Show k+1 cents can be obtained using 2-cent and 5-cent coins (only) as well.

   Basically there are two cases to go from k to k+1.

   - Case 1: If n cents include at least two 2-cent coins, to make k+1 cents we can remove two 2-cent coins and add one 5-cent coin.
   - Case 2: If n cents include at least one 5-cent coin, to make k+1 cents we can remove one 5-cent coin and add three 2-cent coins

   The configuration of the coins for k cents (k >= 5) is always either case 1 or 2. More specifically, having 4 or more cents implies:

   - There are at least two 2-cent coins; or
   - There is at least one 5-cent coin.

   Therefore, when we have k cents, we can always make k+1 cents using 2-cent and 5-cent coins only… (B)
   By (A) and (B), we can conclude that the statement is true (for all integers n >= 1). QED.