Lecture Note #3 (Mathematical Induction)
Exercises

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3) Strong form of Mathematical Induction

#1. [Example 5.4.2, p. 270] Define a sequence s_0, s_1, s_2,.. as follows

$s_0 = 0$, $s_1 = 4$, $s_k = 6s_{k-1} – 5s_{k-2}$ for all integers k >= 2.

Actually the whole proof is shown in the textbook (p. 270-271). You can see it there.
But since I went over this in the class, I type what I wrote (or close to it) here.

a) Find the first four terms
   - s0 = 0
   - s1 = 4
   - s2 = 6*s1 – 5*s0 = 6*4 – 5*0 = 24
   - s3 = 6*s2 – 5*s1 = 6*24 – 5*4 = 144 – 20 = 124

b) We are given that $s_n = 5^n – 1$. Prove true.

Proof: By induction on n. Let $P(n) = 5^n – 1$ (for all n >= 0).

1) Basis step (n = 0 and n = 1):
   When n = 0, $s_0 = 0$ (as given) and $P(0) = 5^0 – 1 = 1 – 1 = 0$.
   When n = 1, $s_0 = 4$ (as given) and $P(1) = 5^1 – 1 = 5 - 1 = 4$.
   Therefore, $s_0 = P(0)$ and $s_1 = P(1)$ … (A)

2) Inductive step:
   [Inductive hypothesis] Assume **s_i = P(i)** for all integer 0 <= i <= k, where k >= 1, that is,
   s_i = $6s_{i-1} – 5s_{i-2}$ **= P(i)** = 5^i - 1.
   [Inductive statement] Show **s_{k+1} = P(k+1)**, that is, **s_{k+1} =** $6s_k – 5s_{k-1}$ **= P(k+1)** = 5^{k+1} - 1.

$$
\begin{aligned}
&LHS \\
&= s_{k+1} \\
&= 6 \cdot s_k - 5 \cdot s_{k-1} \\
&= 6 \cdot [5^k - 1] - 5 \cdot [5^{k-1} - 1] \\
&= 6 \cdot 5^k - 6 - 5 \cdot 5^{k-1} + 5 \\
&= 6 \cdot 5^k - 1 - 5^k \\
&= 5 \cdot 5^k - 1 \\
&= 5^{k+1} - 1 \\
&= P(k+1) \\
&= RHS
\end{aligned}
$$

Therefore, e_{k+1} = P(k+1) … (B)
By (A) and (B), we can conclude that the statement is true (for all integers n >= 0). QED.

#2. Write the Strong Mathematical Induction version of the problem given earlier, "For all integer n >= 4, n cents can be obtained by using 2-cent and 5-cent coins." Note the basis steps should prove P(4) and P(5).

Proof:

1) Basis step (n = 4 and n = 5):
When n = 4, we can make 4 cents by two 2-cent coins.
When n = 5, we can make 5 cents by one 5-cent coin. … (A)

2) Inductive step:
[Inductive hypothesis] Assume n >= 6, and **k** cents can be obtained using 2-cent and 5-cent coins (only), for all **k**, 4 <= **k** <= n.
By the inductive hypothesis, we can make n-2 cents (by taking out one 2-cent coin). So we can add a 2-cent coin to make n cents. … (B)

By (A) and (B), we can conclude that the statement is true (for all integers n >= 1). QED.

#3. [Section 5.4, Exercise #5, p. 277] Suppose that e_0, e_1, e_2,… is a sequence defined as follows.

e_0 = 12, e_1 = 29
$e_k = 5e_{k-1} – 6e_{k-2}$ for all integers k >= 2.

Prove that $e_n = 5*3^n + 7*2^n$ for all integer n >= 0. Use Strong Mathematical Induction.

Actually the whole proof is shown at the back of the textbook (p. A-38). You can see it there.
But since I went over this in the class, I type what I wrote (or close to it) here.

Proof: Let $P(n) = 5*3^n + 7*2^n$ for all integer n >= 0. We show the equivalency between the sequence e_n and P(n), that is, $\mathbf{e_n} = 5e_{n-1} – 6e_{n-2}$ **= P(n)** $= 5*3^n + 7*2^n$.

3) Basis step (n = 0 and n = 1):
When n = 0, e_0 = 12 (as given) and $P(0) = 5*3^0 + 7*2^0 = 5 + 7 = 12$.
When n = 1, e_0 = 29 (as given) and $P(1) = 5*3^1 + 7*2^1 = 15 + 14 = 29$.
Therefore, e_0 = P(0) and e_1 = P(1) … (A)

4) Inductive step:
[Inductive hypothesis] Assume $\mathbf{e_i}$ **= P(i)** for all integer 0 <= i <= k, where k >= 1, that is,
$\mathbf{e_i} = 5e_{i-1} – 6e_{i-2}$ **= P(i)** $= 5*3^i + 7*2^i$.
[Inductive statement] Show $\mathbf{e_{k+1}}$ **= P(k+1)**, that is, $\mathbf{e_{k+1}} = 5e_k – 6e_{k-1}$ **= P(k+1)** $= 5*3^{k+1} + 7*2^{k+1}$.
LHS
$= e_{k+1}$
$= 5 \cdot e_k - 6 \cdot e_{k-1}$

$= 5 \cdot [5 \cdot 3^k + 7 \cdot 2^k] - 6 \cdot [5 \cdot 3^{k-1} + 7 \cdot 2^{k-1}]$ ... by inductive hypothesis
$= 5 \cdot 5 \cdot 3^k + 5 \cdot 7 \cdot 2^k - 6 \cdot [5 \cdot 3^{k-1} + 7 \cdot 2^{k-1}]$
$= 5 \cdot 5 \cdot 3^k + 5 \cdot 7 \cdot 2^k - 2 \cdot 3 \cdot 5 \cdot 3^{k-1} - 2 \cdot 3 \cdot 7 \cdot 2^{k-1}$
$= 5 \cdot 5 \cdot 3^k + 5 \cdot 7 \cdot 2^k - 2 \cdot 5 \cdot 3^k - 3 \cdot 7 \cdot 2^k$
$= 3 \cdot 5 \cdot 3^k + 2 \cdot 7 \cdot 2^k$
$= 5 \cdot 3^{k+1} + 7 \cdot 2^{k+1}$
$= P(k+1)$
$= RHS$
Therefore, e_{k+1} = P(k+1) … (B)
By (A) and (B), we can conclude that the statement is true (for all integers n >= 0). QED.