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// Sample Solutions to Lecture Note #1 Exercise Questions
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1. Lecture note #1, Proving quantified statement, 2.c.

Proposition: For some real number x, x > 5 and x < 10.

Proof:
The proposition is true. Take x = 7. Then we have 7 > 5 and 7 < 10. QED.

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2. Lecture note #1, Proving quantified statement, 2.d.

Proposition: For some real number x, x > 5 and x < 4.

Proof (simple version):
The proposition is false. By assumption, x > 5. Here, values < 4 are not included in that range because they are < 5. Therefore, it's impossible for x to satisfy x < 4.
Therefore, there is no number such that x > 5 and x < 4. QED.

Proof (detailed version):
The proposition is false. To prove, we show the negation is true.
By applying DeMorgan's Laws, the negation is "For all real number x, x <= 5 or x >= 4." This is true because any real number fall in either range (<=5 or >=4). Since the negation is true, the original statement was false. QED.

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3. Lecture note #1, Multiple quantifiers.

a) Forall x, Forall y. x^2 + 2y > 4.

Proof:
This statement is false. A counterexample is x = 1 and y = 1. Then we have x^2 + 2y = 1^2 + 2(1) = 1 + 2 = 3, which is NOT > 4. The fore the proposition is false. QED.

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b) Thereexists x, Thereexists y. x^2 + 2y > 4.

Proof:
This statement is true. Take x = 0 and y = 3. Then we have x^2 + 2y = 0^2 + 2(3) = 6, which is > 4. Therefore the statement is true. QED.

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USFUL NOTE:

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Forall x, Thereexists y. P(x,y) -- e.g. "Everybody loves somebody."
                                        "Every bird has a beak."
                                        (Logic game)
Thereexists x, Forall y. P(x,y) -- e.g. "Somebody loves everybody."
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c) Forall x, Thereexists y. x^2 + 2y > 4.

Proof:
The statement is true. Since x is a real number, x^2 is always >= 0.
Therefore for any x, we can take a value for y such that 2y > 4 - x^2, which gives y > (4 - x^2)/2. QED.

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d) Thereexists x, Forall y. x^2 + 2y > 4.

Proof (simple version):
The statement is false. Since y is any real number, its value, or 2y, could be anywhere between negative infinity and positive infinity. Therefore, there

is no _one_ x value that makes x^2 + 2y to be > 4. QED.

Proof (detailed version):
The statement is false. We show that the negation is true --- Forall x, Thereexists y. x^2 + 2y <= 4.
Since x is a real number, x^2 is always >= 0. Therefore for any x, we can take a value for y such that 2y <= 4 - x^2, which gives y <= (4 - x^2)/2. QED.

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e) Forall x, Forall y, if x < y, then x^2 + 2y > 4.

Proof:
The statement is false. The counterexample is x = 0 and y = 1.
It is indeed the case where x = 0 < 1 = y, but x^2 + 2y = 1 + 2 = 3 <= 4, which contradicts with the given conclusion of x^2 + 2y > 4. QED.

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f) Thereexists x, Thereexists y, if x < y, then x^2 + 2y > 4.

Proof:
The statement is true. Take x = 2 and y = 3. Then we get
x^2 + 2y = 4 + 6 = 10 which is > 4. Therefore the statement is true. QED.

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g) Forall x, Thereexists y, if x < y, then x^2 + 2y > 4.

Proof:
The statement is true. Since x is a real number, x^2 is always >= 0.
Therefore for any x, we can take a value for y which is greater than x (i.e., y > x), which also makes x^2 + 2y greater than 4 (i.e., x^2 + 2y > 4). QED.

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h) Thereexists x, Forall y, if x < y, then x^2 + 2y > 4.

Proof:
The statement is false. We show the negation is true -- Forall x, Thereexists y, such that x < y and x^2 + 2y <= 4. Since x is a real number, x^2 is always >= 0. Therefore for any x, we can take a value for y which is greater than x (i.e., y > x), which also makes x^2 + 2y greater than 4 (i.e., x^2 + 2y > 4).
QED.