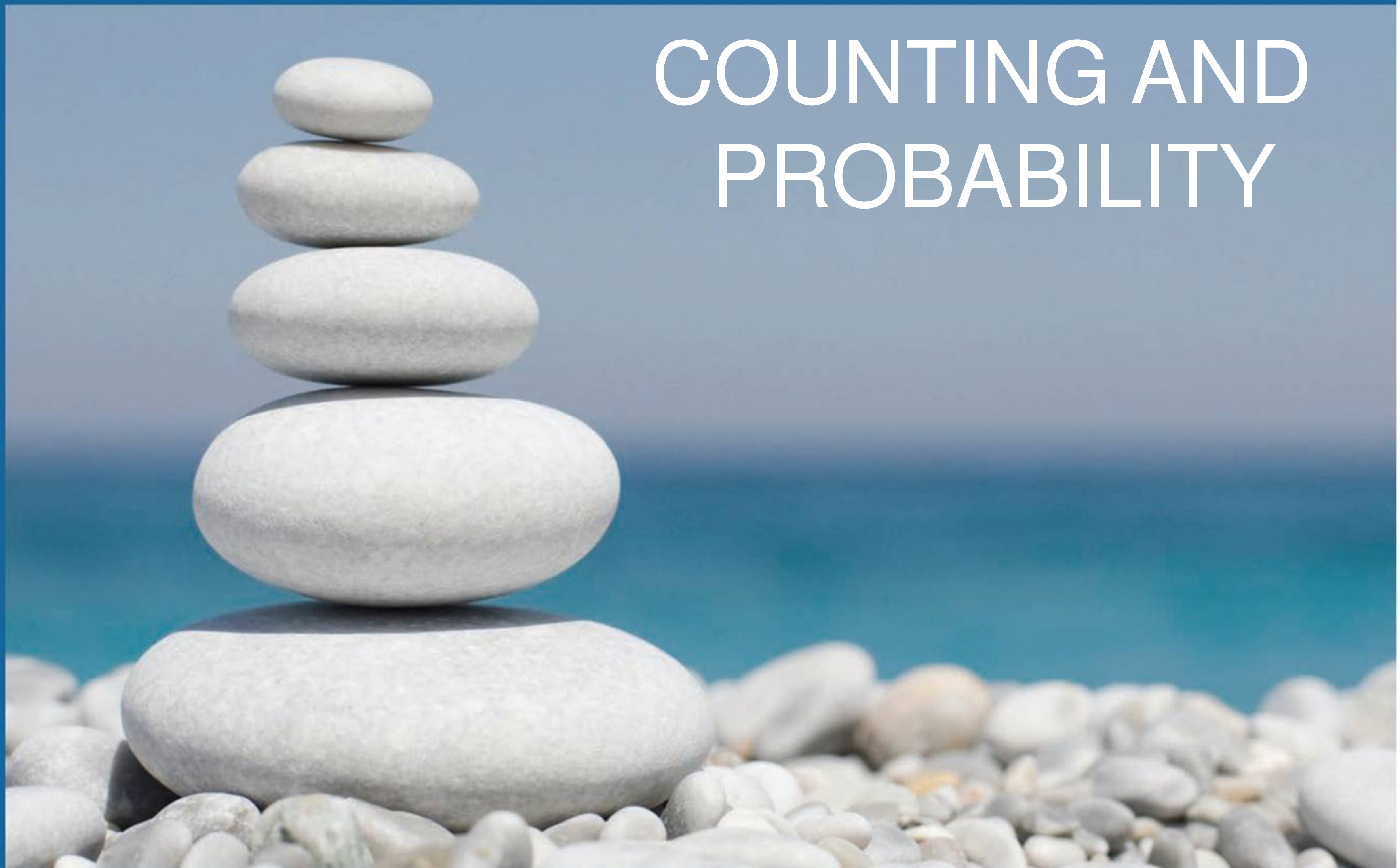
CHAPTER 9

COUNTING AND PROBABILITY



SECTION 9.7

Pascal’s Formula and the Binomial Theorem

Pascal’s Formula and the Binomial Theorem

In this section we derive several formulas for values of $\binom{n}{r}$. The most important is Pascal’s formula, which is the basis for Pascal’s triangle and is a crucial component of one of the proofs of the binomial theorem.

Example 1 – *Values of* $\binom{n}{n}, \binom{n}{n-1}, \binom{n}{n-2}$

Think of Theorem 9.5.1 as a general template.

Theorem 9.5.1

The number of subsets of size r (or r-combinations) that can be chosen from a set of n elements, $\binom{n}{r}$, is given by the formula

$$\binom{n}{r} = \frac{P(n, r)}{r!} \qquad \text{first version}$$

or, equivalently,

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} \qquad \text{second version}$$

where n and r are nonnegative integers with $r \le n$.

Example 1 – *Values of* $\binom{n}{n}, \binom{n}{n-1}, \binom{n}{n-2}$ cont'd

Regardless of what nonnegative numbers are placed in the boxes, if the number in the lower box is no greater than the number in the top box, then

$$\binom{\bullet}{\bullet} = \frac{\bullet\,!}{\bullet\,!(\bullet - \bullet)!}.$$

Use Theorem 9.5.1 to show that for all integers *n* ≥ *0*,

$$\binom{n}{n} = 1 \qquad 9.7.1$$

$$\binom{n}{n-1} = n, \quad \text{if } n \geq 1 \qquad 9.7.2$$

$$\binom{n}{n-2} = \frac{n(n-1)}{2}, \quad \text{if } n \geq 2. \qquad 9.7.3$$

Example 1 – *Solution*

$$\binom{n}{n} = \frac{n!}{n!(n-n)!}$$

$$= \frac{1}{0!}$$

$$= 1 \qquad \text{since } 0! = 1 \text{ by definition}$$

$$\binom{n}{n-1} = \frac{n!}{(n-1)!(n-(n-1))!}$$

$$= \frac{n \cdot (n-1)!}{(n-1)!(n-n+1)!}$$

$$= \frac{n}{1}$$

$$= n$$

Example 1 – *Solution*

cont'd

$$\binom{n}{n-2} = \frac{n!}{(n-2)!(n-(n-2))!}$$

$$= \frac{n \cdot (n-1) \cdot \cancel{(n-2)!}}{\cancel{(n-2)!}2!}$$

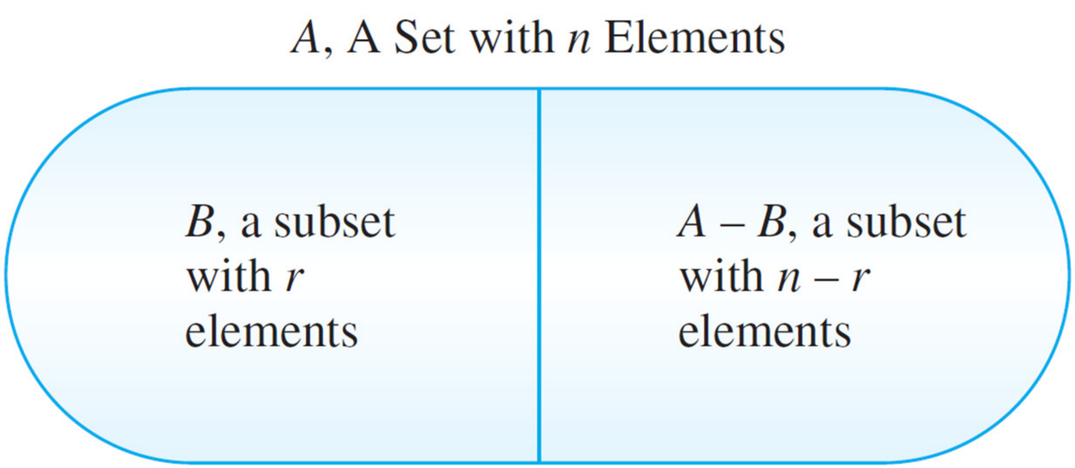
$$= \frac{n(n-1)}{2}$$

Example 2 – $\binom{n}{r} = \binom{n}{n-r}$

Deduce the formula $\binom{n}{r} = \binom{n}{n-r}$ for all nonnegative integers n and r with $r \leq n$, by interpreting it as saying that a set A with n elements has exactly as many subsets of size r as it has subsets of size $n - r$.

Example 2 – *Solution*

Observe that any subset of size r can be specified either by saying which r elements lie in the subset or by saying which $n - r$ elements lie outside the subset.



Any subset B with r elements completely determines a subset, $A - B$, with $n - r$ elements.

Example 2 – *Solution*

cont'd

Suppose A has k subsets of size r : $B_1, B_2, \ldots, B_k$.

Then each B_i can be paired up with exactly one set of size $n - r$, namely its complement $A - B_i$ as shown below.

Subsets of Size r		Subsets of Size $n - r$
B_1	$\longleftrightarrow$	$A - B_1$
B_2	$\longleftrightarrow$	$A - B_2$
$\vdots$		$\vdots$
B_k	$\longleftrightarrow$	$A - B_k$

Example 2 – *Solution*

cont'd

All subsets of size r are listed in the left-hand column, and all subsets of size $n - r$ are listed in the right-hand column.

The number of subsets of size r equals the number of subsets of size $n - r$, and so $\binom{n}{r} = \binom{n}{n-r}$.

Pascal’s Formula and the Binomial Theorem

The type of reasoning used in this example is called *combinatorial*, because it is obtained by counting things that are combined in different ways.

A number of theorems have both combinatorial proofs and proofs that are purely algebraic.

Pascal's Formula

Pascal's Formula

It relates the value of $\binom{n+1}{r}$ to the values of $\binom{n}{r-1}$ and $\binom{n}{r}$. Specifically, it says that

$$\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}$$

whenever n and r are positive integers with $r \leq n$.

This formula makes it easy to compute higher combinations in terms of lower ones: If all the values of $\binom{n}{r}$ are known, then the values of $\binom{n+1}{r}$ can be computed for all r such that $0 < r \leq n$.

Pascal’s Formula

Pascal’s triangle, shown in Table 9.7.1, is a geometric version of Pascal’s formula.

It has generally been known as Pascal’s triangle.

n \ r	0	1	2	3	4	5	…	$r-1$	r	…
0	1							.	.	…
1	1	1						.	.	…
2	1	2	1					.	.	…
3	1	3	3	1				.	.	…
4	1	4	6	+ 4	1			.	.	…
5	1	5	10	= 10	5	1		.	.	…
⋮	⋮	⋮	⋮	⋮	⋮	⋮		⋮	⋮	⋮
n	$\binom{n}{0}$	$\binom{n}{1}$	$\binom{n}{2}$	$\binom{n}{3}$	$\binom{n}{4}$	$\binom{n}{5}$	…	$\binom{n}{r-1}$ +	$\binom{n}{r}$	…
$n+1$	$\binom{n+1}{0}$	$\binom{n+1}{1}$	$\binom{n+1}{2}$	$\binom{n+1}{3}$	$\binom{n+1}{4}$	$\binom{n+1}{5}$	…		= $\binom{n+1}{r}$	…
.		.	.	.	.	.		.	.	…
.		.	.	.	.	.		.	.	…
.		.	.	.	.	.		.	.	…

Pascal’s Triangle for $\left(\text{Values of } \binom{n}{r}\right)$

Table 9.7.1

Pascal's Formula

Each entry in the triangle is a value of $\binom{n}{r}$.

Pascal's formula translates into the fact that the entry in row $n + 1$, column r equals the sum of the entry in row n, column $r - 1$ plus the entry in row n, column r.

That is, the entry in a given interior position equals the sum of the two entries directly above and to the above left. The left-most and right-most entries in each row are 1 because $\binom{n}{n} = 1$ by Example 1 and $\binom{n}{0} = 1$.

Example 3 – *Calculating* $\binom{n}{r}$ *Using Pascal's Triangle*

Use Pascal's triangle to compute the values of

$$\binom{6}{2} \quad \text{and} \quad \binom{6}{3}.$$

Solution:

By construction, the value in row *n*, column *r* of Pascal's triangle is the value of $\binom{n}{r}$, for every pair of positive integers *n* and *r* with $r \leq n$.

By Pascal's formula, $\binom{n+1}{r}$ can be computed by adding together $\binom{n}{r-1}$ and $\binom{n}{r}$, which are located directly above and above left of $\binom{n+1}{r}$.

Example 3 – *Solution*

cont'd

Thus,

$$\binom{6}{2} = \binom{5}{1} + \binom{5}{2} = 5 + 10 = 15$$

and

$$\binom{6}{3} = \binom{5}{2} + \binom{5}{3} = 10 + 10 = 20.$$

Pascal's Formula

Pascal's formula can be derived by two entirely different arguments. One is algebraic; it uses the formula for the number of *r*-combinations obtained in Theorem 9.5.1.

The other is combinatorial; it uses the definition of the number of *r*-combinations as the number of subsets of size *r* taken from a set with a certain number of elements.

Theorem 9.7.1 Pascal's Formula

Let n and r be positive integers and suppose $r \leq n$. Then

$$\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}.$$

Example 4 – *Deriving New Formulas from Pascal's Formula*

Use Pascal's formula to derive a formula for $\binom{n+2}{r}$ in terms of values of $\binom{n}{r}$, $\binom{n}{r-1}$, and $\binom{n}{r-2}$.

Assume *n* and *r* are nonnegative integers and $2 \le r \le n$.

Solution:

By Pascal's formula,

$$\binom{n+2}{r} = \binom{n+1}{r-1} + \binom{n+1}{r}.$$

Example 4 – *Solution*

cont'd

Now apply Pascal's formula to $\binom{n+1}{r-1}$ and $\binom{n+1}{r}$ and substitute into the above to obtain

$$\binom{n+2}{r} = \left[\binom{n}{r-2} + \binom{n}{r-1}\right] + \left[\binom{n}{r-1} + \binom{n}{r}\right].$$

Combining the two middle terms gives

$$\binom{n+2}{r} = \binom{n}{r-2} + 2\binom{n}{r-1} + \binom{n}{r}$$

for all nonnegative integers *n* and *r* such that $2 \le r \le n$.

The Binomial Theorem

The Binomial Theorem

In algebra a sum of two terms, such as $a + b$, is called a **binomial**.

The *binomial theorem* gives an expression for the powers of a binomial $(a + b)^n$, for each positive integer n and all real numbers a and b.

Theorem 9.7.2 Binomial Theorem

Given any real numbers a and b and any nonnegative integer n,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

$$= a^n + \binom{n}{1} a^{n-1} b^1 + \binom{n}{2} a^{n-2} b^2 + \cdots + \binom{n}{n-1} a^1 b^{n-1} + b^n.$$

The Binomial Theorem

Note that the second expression equals the first because $\binom{n}{0} = 1$ and $\binom{n}{n} = 1$, for all nonnegative integers *n*, provided that $b^0 = 1$ and $a^{n-n} = 1$.

• Definition

For any real number a and any nonnegative integer n, the **nonnegative integer powers of *a*** are defined as follows:

$$a^n = \begin{cases} 1 & \text{if } n = 0 \\ a \cdot a^{n-1} & \text{if } n > 0 \end{cases}$$

The Binomial Theorem

If n and r are nonnegative integers and $r \leq n$, then $\binom{n}{r}$ is called a **binomial coefficient** because it is one of the coefficients in the expansion of the binomial expression $(a + b)^n$.

Example 5 – *Substituting into the Binomial Theorem*

Expand the following expressions using the binomial theorem:

a. $(a + b)^5$ **b.** $(x - 4y)^4$

Solution:

a. $$(a+b)^5 = \sum_{k=0}^{5} \binom{5}{k} a^{5-k} b^k$$

$$= a^5 + \binom{5}{1} a^{5-1} b^1 + \binom{5}{2} a^{5-2} b^2 + \binom{5}{3} a^{5-3} b^3 + \binom{5}{4} a^{5-4} b^4 + b^5$$

$$= a^5 + 5a^4 b + 10a^3 b^2 + 10a^2 b^3 + 5ab^4 + b^5$$

Example 5 – *Solution*

cont'd

b. Observe that $(x - 4y)^4 = (x + (-4y))^4$.

So let *a* = *x* and *b* = (–4*y*), and substitute into the binomial theorem.

$$(x - 4y)^4 = \sum_{k=0}^{4} \binom{4}{k} x^{4-k}(-4y)^k$$

$$= x^4 + \binom{4}{1} x^{4-1}(-4y)^1 + \binom{4}{2} x^{4-2}(-4y)^2 + \binom{4}{3} x^{4-3}(-4y)^3 + (-4y)^4$$

$$= x^4 + 4x^3(-4y) + 6x^2(16y^2) + 4x^1(-64y^3) + (256y^4)$$

$$= x^4 - 16x^3y + 96x^2y^2 - 256xy^3 + 256y^4$$

Example 7 – *Using a Combinatorial Argument to Derive the Identity*

According to Theorem 6.3.1, a set with *n* elements has 2^n subsets.

> **Theorem 6.3.1**
>
> For all integers $n \geq 0$, if a set X has n elements, then $\mathscr{P}(X)$ has 2^n elements.

Apply this fact to give a combinatorial argument to justify the identity

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \cdots + \binom{n}{n} = 2^n.$$

Example 7 – *Solution*

Suppose *S* is a set with *n* elements. Then every subset of *S* has some number of elements *k*, where *k* is between 0 and *n*.

It follows that the total number of subsets of S, $N(\mathscr{P}(S))$, can be expressed as the following sum:

$$\begin{bmatrix}\text{number of} \\ \text{subsets} \\ \text{of } S\end{bmatrix} = \begin{bmatrix}\text{number of} \\ \text{subsets of} \\ \text{size } 0\end{bmatrix} + \begin{bmatrix}\text{number of} \\ \text{subsets of} \\ \text{size } 1\end{bmatrix} + \cdots + \begin{bmatrix}\text{number of} \\ \text{subsets of} \\ \text{size } n\end{bmatrix}.$$

Now the number of subsets of size *k* of a set with *n* elements is $\binom{n}{k}$.

Example 7 – *Solution*

cont'd

Hence,

the number of subsets of $S = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}$

But by Theorem 6.3.1, *S* has 2^n subsets. Hence

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \cdots + \binom{n}{n} = 2^n.$$

Example 8 – *Using the Binomial Theorem to Simplify a Sum*

Express the following sum in **closed form** (without using a summation symbol and without using an ellipsis · · ·):

$$\sum_{k=0}^{n} \binom{n}{k} 9^k$$

Solution:

When the number 1 is raised to any power, the result is still 1. Thus

$$\begin{aligned}\sum_{k=0}^{n} \binom{n}{k} 9^k &= \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} 9^k \\ &= (1+9)^n \quad \text{by the binomial theorem with } a = 1 \text{ and } b = 9 \\ &= 10^n.\end{aligned}$$