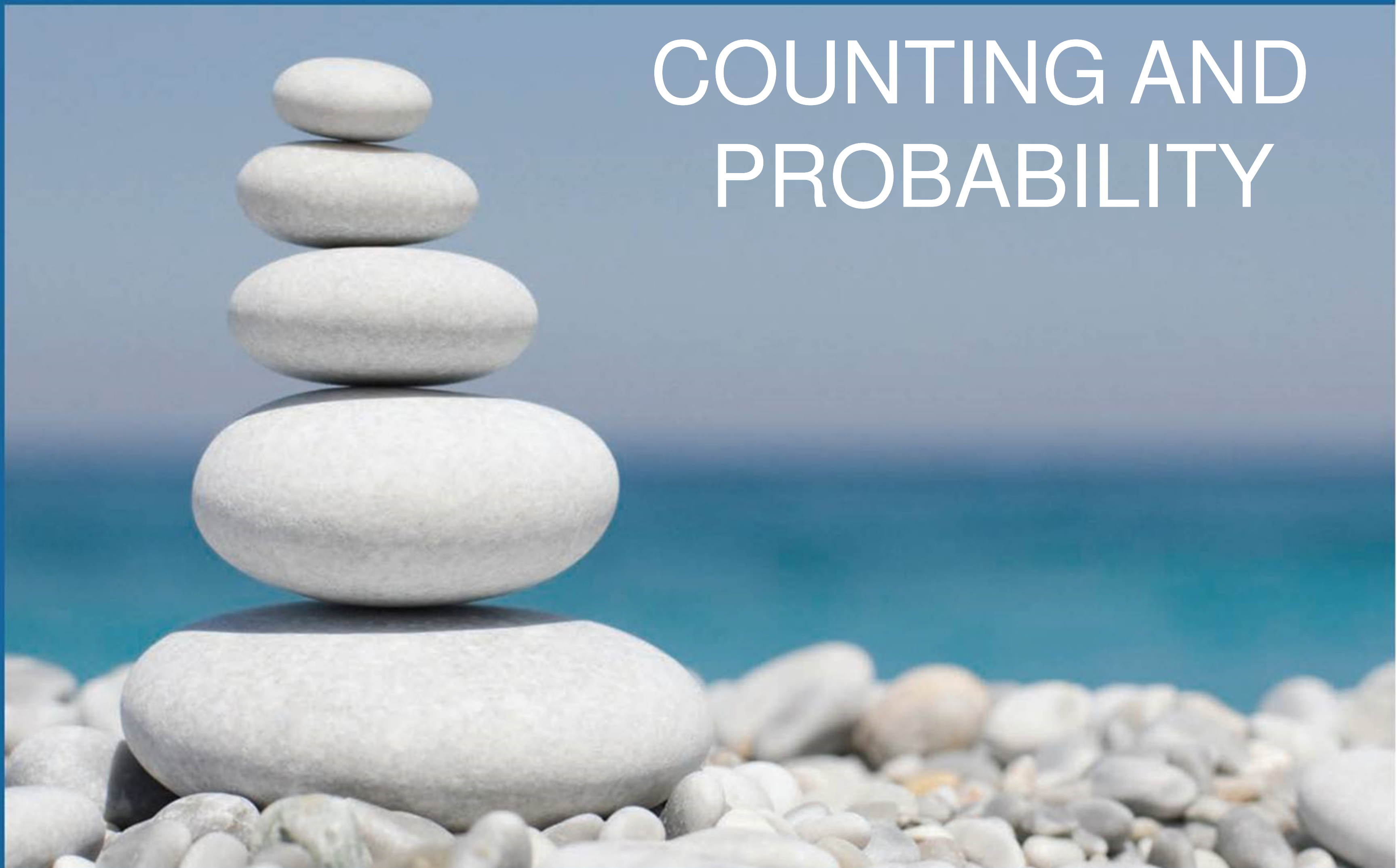
CHAPTER 9

COUNTING AND PROBABILITY



SECTION 9.5

Counting Subsets of a Set: Combinations

Counting Subsets of a Set: Combinations

Given a set S with n elements, how many subsets of size r can be chosen from S?

The number of subsets of size r that can be chosen from S equals the number of subsets of size r that S has.

Each individual subset of size r is called an *r-combination* of the set.

Counting Subsets of a Set: Combinations

• Definition

Let n and r be nonnegative integers with $r \le n$. An ***r*-combination** of a set of n elements is a subset of r of the n elements. As indicated in Section 5.1, the symbol

$$\binom{n}{r},$$

which is read "n choose r," denotes the number of subsets of size r (r-combinations) that can be chosen from a set of n elements.

We have known that calculators generally use symbols like $C(n, r)$, ${}_nC_r$, $C_{n,r}$, or ${}^nC_r$ instead of $\binom{n}{r}$.

Example 1 – 3-*Combinations*

Let S = {Ann, Bob, Cyd, Dan}. Each committee consisting of three of the four people in S is a 3-combination of S.

a. List all such 3-combinations of S.

b. What is $\binom{4}{3}$?

Solution:

a. Each 3-combination of S is a subset of S of size 3. But each subset of size 3 can be obtained by leaving out one of the elements of S.

The 3-combinations are

{Bob, Cyd, Dan} leave out Ann

Example 1 – *Solution*

cont'd

{Ann, Cyd, Dan} leave out Bob

{Ann, Bob, Dan} leave out Cyd

{Ann, Bob, Cyd} leave out Dan.

b. Because $\binom{4}{3}$ is the number of 3-combinations of a set with four elements, by part (**a**), $\binom{4}{3} = 4$.

Counting Subsets of a Set: Combinations

There are two distinct methods that can be used to select r objects from a set of n elements. In an **ordered selection**, it is not only what elements are chosen but also the order in which they are chosen that matters.

Two ordered selections are said to be the same if the elements chosen are the same and also if the elements are chosen in the same order. An ordered selection of r elements from a set of n elements is an r-permutation of the set.

Counting Subsets of a Set: Combinations

In an **unordered selection**, on the other hand, it is only the identity of the chosen elements that matters. Two unordered selections are said to be the same if they consist of the same elements, regardless of the order in which the elements are chosen.

An unordered selection of *r* elements from a set of *n* elements is the same as a subset of size *r* or an *r*-combination of the set.

Example 2 – *Unordered Selections*

How many unordered selections of two elements can be made from the set {0, 1, 2, 3}?

Solution:

An unordered selection of two elements from {0, 1, 2, 3} is the same as a 2-combination, or subset of size 2, taken from the set.

These can be listed systematically:

{0, 1}, {0, 2}, {0, 3} subsets containing 0

{1, 2}, {1, 3} subsets containing 1 but not already listed

Example 2 – *Solution*

cont'd

{2, 3} subsets containing 2 but not already listed.

Since this listing exhausts all possibilities, there are six subsets in all.

Thus $\binom{4}{2}$ = 6, which is the number of unordered selections of two elements from a set of four.

Counting Subsets of a Set: Combinations

When the values of *n* and *r* are small, it is reasonable to calculate values of $\binom{n}{r}$ using the method of **complete enumeration** (listing all possibilities) illustrated in Examples 1 and 2. But when *n* and *r* are large, it is not feasible to compute these numbers by listing and counting all possibilities.

The general values of $\binom{n}{r}$ can be found by a somewhat indirect but simple method.

An equation is derived that contains $\binom{n}{r}$ as a factor. Then this equation is solved to obtain a formula for $\binom{n}{r}$. The method is illustrated in the next Example.

Example 3 – *Relation between Permutations and Combinations*

Write all 2-permutations of the set {0, 1, 2, 3}. Find an equation relating the number of 2-permutations, *P*(4, 2), and the number of 2-combinations, $\binom{4}{2}$, and solve this equation for $\binom{4}{2}$.

Solution:

According to Theorem 9.2.3, the number of 2-permutations of the set {0, 1, 2, 3} is *P*(4, 2), which equals

$$\frac{4!}{(4-2)!} = \frac{4 \cdot 3 \cdot \cancel{2} \cdot \cancel{1}}{\cancel{2} \cdot \cancel{1}} = 12.$$

Example 3 – *Solution*

cont'd

Now the act of constructing a 2-permutation of {0, 1, 2, 3} can be thought of as a two-step process:

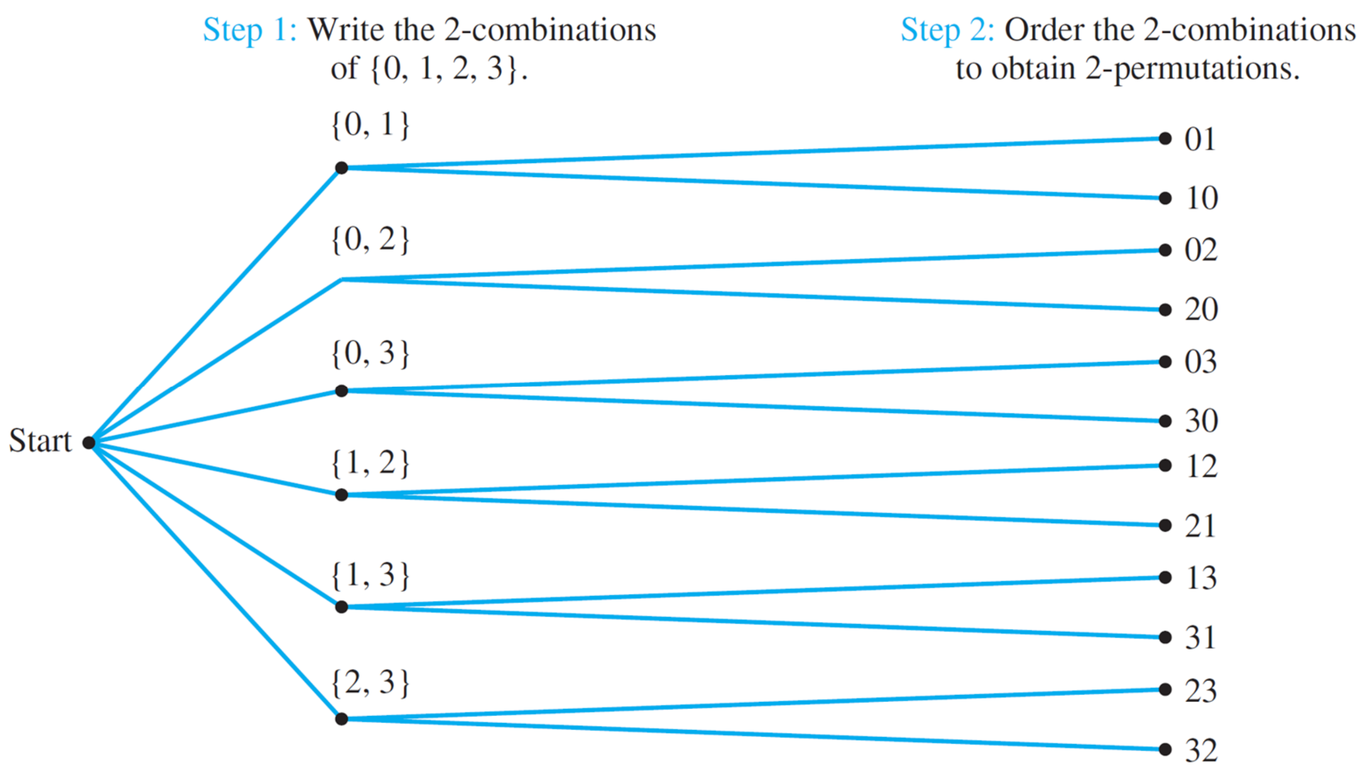
Step 1: Choose a subset of two elements from {0, 1, 2, 3}.

Step 2: Choose an ordering for the two-element subset.

Example 3 – *Solution*

cont'd

This process can be illustrated by the possibility tree shown in Figure 9.5.1.



Relation between Permutations and Combinations

Figure 9.5.1

Example 3 – *Solution*

cont'd

The number of ways to perform step 1 is $\binom{4}{2}$, the same as the number of subsets of size 2 that can be chosen from {0, 1, 2, 3}.

The number of ways to perform step 2 is 2!, the number of ways to order the elements in a subset of size 2.

Because the number of ways of performing the whole process is the number of 2-permutations of the set {0, 1, 2, 3}, which equals *P*(4, 2), it follows from the product rule that

$$P(4, 2) = \binom{4}{2} \cdot 2!.$$

This is an equation that relates $P(4, 2)$ and $\binom{4}{2}$.

Example 3 – *Solution*

cont'd

Solving the equation for $\binom{4}{2}$ gives

$$\binom{4}{2} = \frac{P(4, 2)}{2!}$$

We know that $P(4, 2) = \frac{4!}{(4-2)!}$.

Hence, substituting yields

$$\binom{4}{2} = \frac{\frac{4!}{(4-2)!}}{2!}$$

$$= \frac{4!}{2!(4-2)!}$$

$$= 6.$$

Counting Subsets of a Set: Combinations

The reasoning used in Example 3 applies in the general case as well.

Theorem 9.5.1

The number of subsets of size r (or r-combinations) that can be chosen from a set of n elements, $\binom{n}{r}$, is given by the formula

$$\binom{n}{r} = \frac{P(n, r)}{r!} \qquad \text{first version}$$

or, equivalently,

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} \qquad \text{second version}$$

where n and r are nonnegative integers with $r \leq n$.

Example 4 – *Calculating the Number of Teams*

Consider again the problem of choosing five members from a group of twelve to work as a team on a special project. How many distinct five-person teams can be chosen?

Solution:

The number of distinct five-person teams is the same as the number of subsets of size 5 (or 5-combinations) that can be chosen from the set of twelve. This number is $\binom{12}{5}$.

By Theorem 9.5.1,

$$\binom{12}{5} = \frac{12!}{5!(12-5)!} = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7!}{(5 \cdot 4 \cdot 3 \cdot 2 \cdot 1) \cdot 7!} = 11 \cdot 9 \cdot 8 = 792.$$

Thus there are 792 distinct five-person teams.

Counting Subsets of a Set: Combinations

The formula for the number of *r*-combinations of a set can be applied in a wide variety of situations. Let us illustrate this in the next example.

Before we begin the next example, a remark on the phrases *at least* and *at most* is in order:

> The phrase **at least *n*** means "*n* or more."
> The phrase **at most *n*** means "*n* or fewer."

For instance, if a set consists of three elements and you are to choose at least two, you will choose two or three; if you are to choose at most two, you will choose none, or one, or two.

Example 7 – *Teams with Members of Two Types*

Suppose the group of twelve consists of five men and seven women.

a. How many five-person teams can be chosen that consist of three men and two women?

b. How many five-person teams contain at least one man?

c. How many five-person teams contain at most one man?

Solution:

a. To answer this question, think of forming a team as a two-step process:

Step 1: Choose the men.

Example 7 – *Solution*

cont'd

Step 2: Choose the women.

There are $\binom{5}{3}$ ways to choose the three men out of the five and $\binom{7}{2}$ ways to choose the two women out of the seven.

Hence, by the product rule,

$$\begin{bmatrix}\text{number of teams of five that} \\ \text{contain three men and two women}\end{bmatrix} = \binom{5}{3}\binom{7}{2} = \frac{5!}{3!2!} \cdot \frac{7!}{2!5!}$$

$$= \frac{7 \cdot 6 \cdot 5 \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot 1}{\cancel{3} \cdot \cancel{2} \cdot 1 \cdot \cancel{2} \cdot 1 \cdot \cancel{2} \cdot 1}$$

$$= 210.$$

Example 7 – *Solution*

cont'd

b. This question can also be answered either by the addition rule or by the difference rule. The solution by the difference rule is shorter and is shown first.

Observe that the set of five-person teams containing at least one man equals the set difference between the set of all five-person teams and the set of five-person teams that do not contain any men.

Example 7 – *Solution*

cont'd

See Figure 9.5.5.

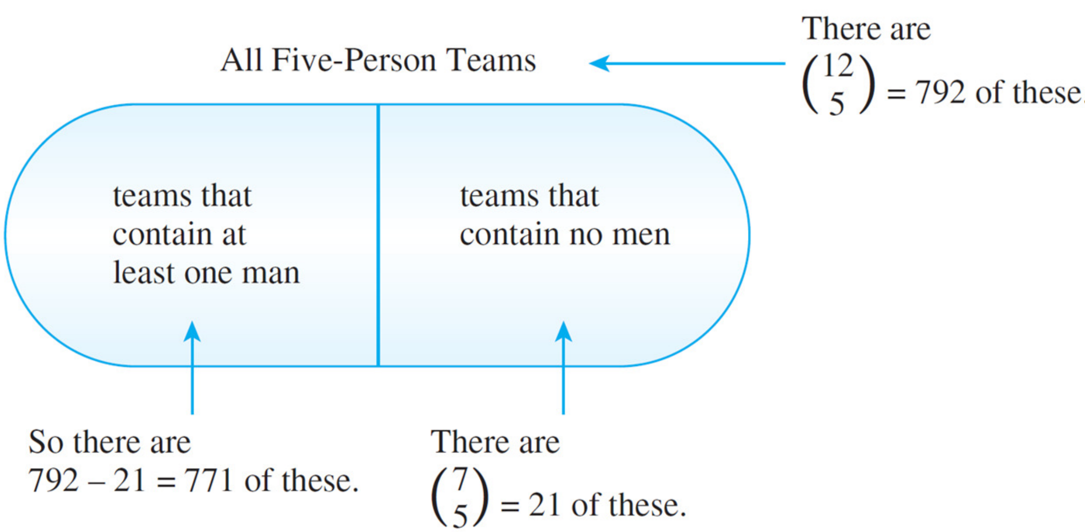


Figure 9.5.5

Example 7 – *Solution*

cont'd

Now a team with no men consists entirely of five women chosen from the seven women in the group, so there are $\binom{7}{5}$ such teams. Also, by Example 4, the total number of five-person teams is $\binom{12}{5}$ = 792.

Hence, by the difference rule,

$$\begin{bmatrix}\text{number of teams}\\ \text{with at least}\\ \text{one man}\end{bmatrix} = \begin{bmatrix}\text{total number}\\ \text{of teams}\\ \text{of five}\end{bmatrix} - \begin{bmatrix}\text{number of teams}\\ \text{of five that do not}\\ \text{contain any men}\end{bmatrix}$$

$$= \binom{12}{5} - \binom{7}{5} = 792 - \frac{7!}{5! \cdot 2!}$$

Example 7 – *Solution*

cont'd

$$= 792 - \frac{7 \cdot \overset{3}{\cancel{6}} \cdot \cancel{5!}}{\cancel{5!} \cdot \cancel{2} \cdot 1} = 792 - 21 = 771.$$

This reasoning is summarized in Figure 9.5.5.

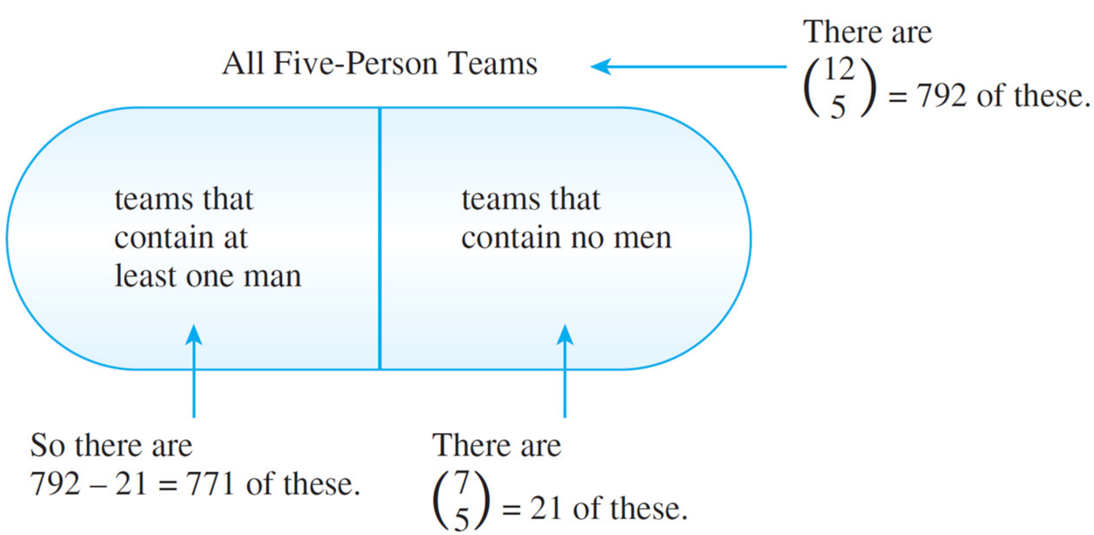
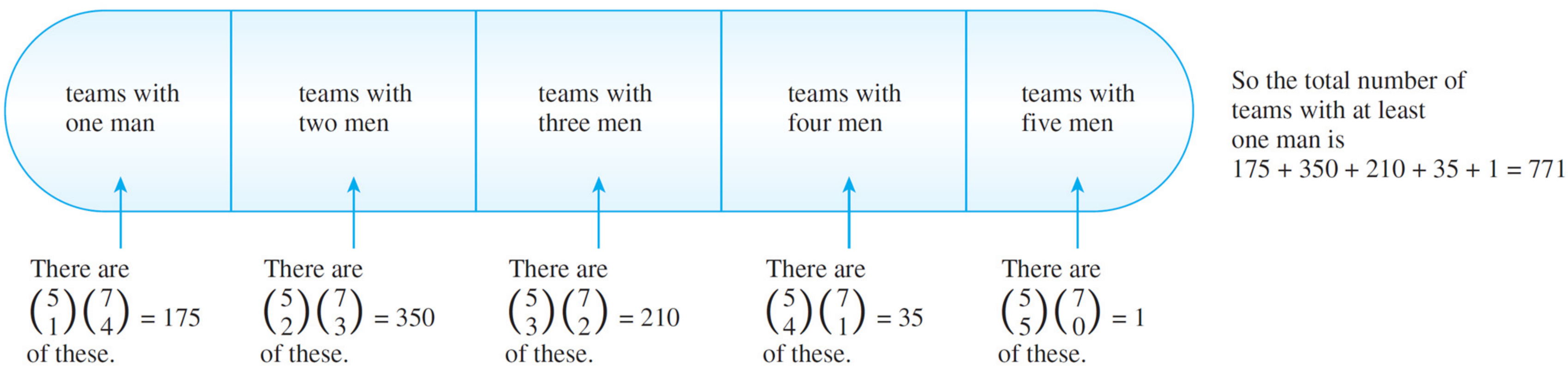


Figure 9.5.5

Example 7 – *Solution*

cont'd

Alternatively, to use the addition rule, observe that the set of teams containing at least one man can be partitioned as shown in Figure 9.5.6.



Teams with At Least One Man

Figure 9.5.6

Example 7 – *Solution*

cont'd

The number of teams in each subset of the partition is calculated using the method illustrated in part (**a**). There are

$\binom{5}{1}\binom{7}{4}$ teams with one man and four women

$\binom{5}{2}\binom{7}{3}$ teams with two men and three women

$\binom{5}{3}\binom{7}{2}$ teams with three men and two women

$\binom{5}{4}\binom{7}{1}$ teams with four men and one woman

Example 7 – *Solution*

cont'd

$$\binom{5}{5}\binom{7}{0} \text{ teams with five men and no women.}$$

Hence, by the addition rule,

$$\begin{bmatrix}\text{number of teams with} \\ \text{at least one man}\end{bmatrix}$$

$$= \binom{5}{1}\binom{7}{4} + \binom{5}{2}\binom{7}{3} + \binom{5}{3}\binom{7}{2} + \binom{5}{4}\binom{7}{1} + \binom{5}{5}\binom{7}{0}$$

$$= \frac{5!}{1!4!}\cdot\frac{7!}{4!3!} + \frac{5!}{2!3!}\cdot\frac{7!}{3!4!} + \frac{5!}{3!2!}\cdot\frac{7!}{2!5!} + \frac{5!}{4!1!}\cdot\frac{7!}{1!6!} + \frac{5!}{5!0!}\cdot\frac{7!}{0!7!}$$

Example 7 – *Solution*

cont'd

$$= \frac{5 \cdot \cancel{4!} \cdot 7 \cdot \cancel{6} \cdot 5 \cdot \cancel{4!}}{\cancel{4!} \cdot \cancel{3 \cdot 2} \cdot \cancel{4!}} + \frac{5 \cdot \overset{2}{\cancel{4}} \cdot \cancel{3!} \cdot 7 \cdot \cancel{6} \cdot 5 \cdot \cancel{4!}}{\cancel{3!} \cdot \cancel{2} \cdot \cancel{4!} \cdot \cancel{3 \cdot 2}} + \frac{5 \cdot \overset{2}{\cancel{4}} \cdot \cancel{3!} \cdot 7 \cdot \overset{3}{\cancel{6}} \cdot \cancel{5!}}{\cancel{2} \cdot \cancel{3!} \cdot \cancel{5!} \cdot \cancel{2}}$$

$$+ \frac{5 \cdot \cancel{4!} \cdot 7 \cdot \cancel{6!}}{\cancel{4!} \cdot \cancel{6!}} + \frac{\cancel{5!} \cdot \cancel{7!}}{\cancel{5!} \cdot \cancel{7!}}$$

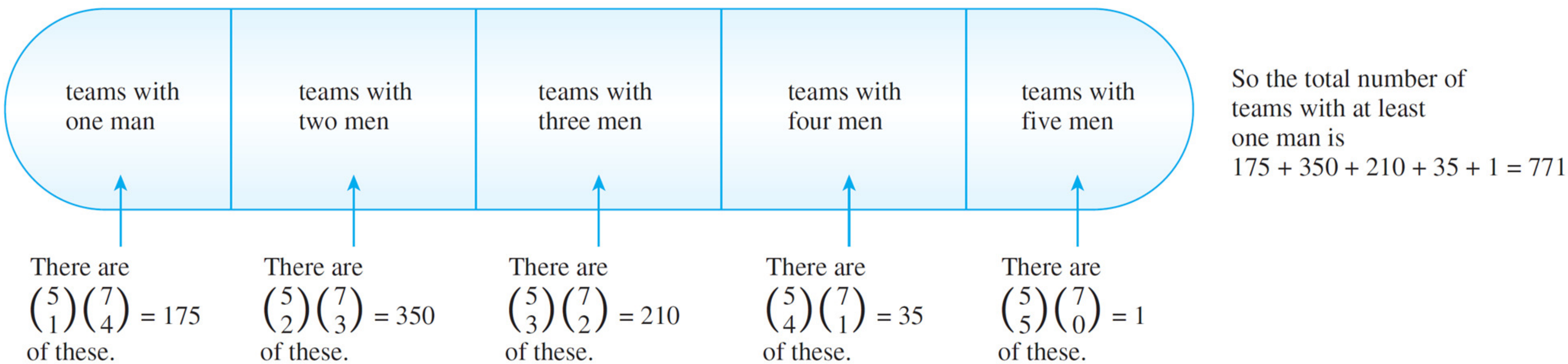
$$= 175 + 350 + 210 + 35 + 1$$

$$= 771.$$

Example 7 – *Solution*

cont'd

This reasoning is summarized in Figure 9.5.6.



Teams with At Least One Man

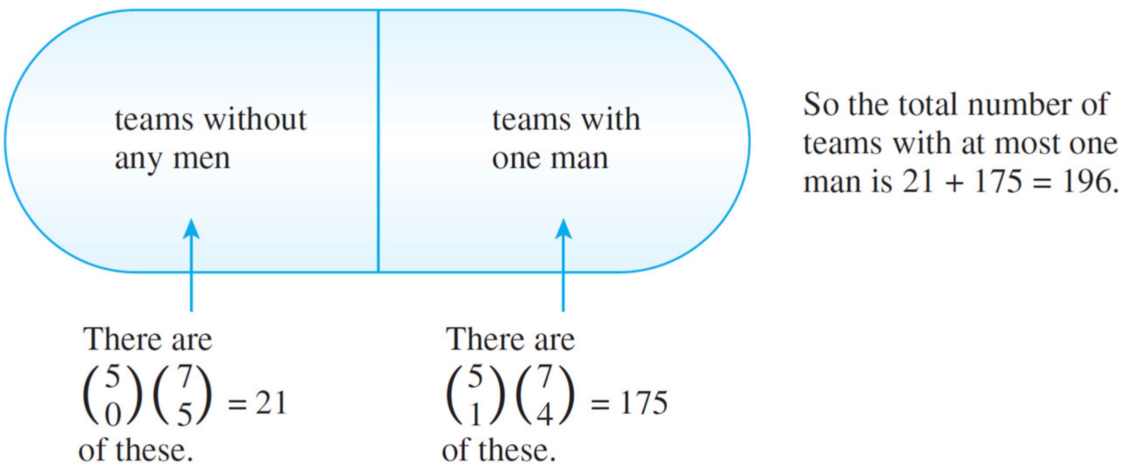
Figure 9.5.6

Example 7 – *Solution*

cont'd

c. As shown in Figure 9.5.7, the set of teams containing at most one man can be partitioned into the set that does not contain any men and the set that contains exactly one man.

Hence, by the addition rule,



Teams with At Most One Man

Figure 9.5.7

Example 7 – *Solution*

cont'd

$$\begin{bmatrix}\text{number of teams} \\ \text{with at} \\ \text{most one man}\end{bmatrix} = \begin{bmatrix}\text{number of} \\ \text{teams without} \\ \text{any men}\end{bmatrix} + \begin{bmatrix}\text{number of} \\ \text{teams with} \\ \text{one man}\end{bmatrix}$$

$$= \binom{5}{0}\binom{7}{5} + \binom{5}{1}\binom{7}{4}$$

$$= 21 + 175$$

$$= 196.$$

This reasoning is summarized in Figure 9.5.7.

Example 10 – *Permutations of a Set with Repeated Elements*

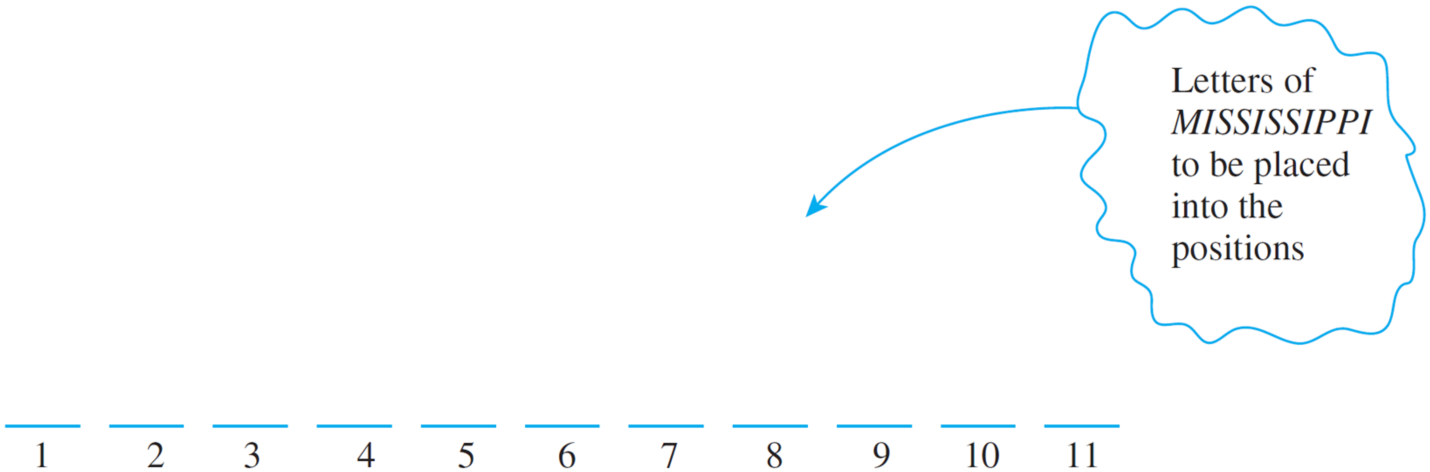
Consider various ways of ordering the letters in the word *MISSISSIPPI*:

IIMSSPISSIP, *ISSSPMIIPIS*, *PIMISSSSIIP*, and so on.

How many distinguishable orderings are there?

Example 10 – *Solution*

Imagine placing the 11 letters of *MISSISSIPPI* one after another into 11 positions.



Because copies of the same letter cannot be distinguished from one another, once the positions for a certain letter are known, then all copies of the letter can go into the positions in any order.

Example 10 – *Solution*

cont'd

It follows that constructing an ordering for the letters can be thought of as a four-step process:

Step 1: Choose a subset of four positions for the *S*'s.

Step 2: Choose a subset of four positions for the *I*'s.

Step 3: Choose a subset of two positions for the *P*'s.

Step 4: Choose a subset of one position for the *M*.

Since there are 11 positions in all, there are $\binom{11}{4}$ subsets of four positions for the *S*'s.

Example 10 – *Solution*

cont'd

Once the four *S*'s are in place, there are seven positions that remain empty, so there are $\binom{7}{4}$ subsets of four positions for the *I*'s. After the *I*'s are in place, there are three positions left empty, so there are $\binom{3}{2}$ subsets of two positions for the *P*'s.

That leaves just one position for the *M*. But 1 = $\binom{1}{1}$. Hence by the multiplication rule,

$$\begin{bmatrix}\text{number of ways to}\\ \text{position all the letters}\end{bmatrix} = \binom{11}{4}\binom{7}{4}\binom{3}{2}\binom{1}{1}$$

$$= \frac{11!}{4!\cancel{7!}}\cdot\frac{\cancel{7!}}{4!\cancel{3!}}\cdot\frac{\cancel{3!}}{2!\cancel{1!}}\cdot\frac{\cancel{1!}}{1!\cancel{0!}}$$

Example 10 – *Solution*

cont'd

$$= \frac{11!}{4! \cdot 4! \cdot 2! \cdot 1!}$$

$$= 34{,}650.$$

Counting Subsets of a Set: Combinations

The reasoning used in this example can be used to derive the following general theorem.

Theorem 9.5.2 Permutations with sets of Indistinguishable Objects

Suppose a collection consists of n objects of which

n_1 are of type 1 and are indistinguishable from each other

n_2 are of type 2 and are indistinguishable from each other

$\vdots$

n_k are of type k and are indistinguishable from each other,

and suppose that $n_1 + n_2 + \cdots + n_k = n$. Then the number of distinguishable permutations of the n objects is

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n-n_1-n_2-\cdots-n_{k-1}}{n_k}$$

$$= \frac{n!}{n_1!\,n_2!\,n_3!\cdots n_k!}.$$

Some Advice about Counting

Some Advice about Counting

Students learning counting techniques often ask, “How do I know what to multiply and what to add? When do I use the multiplication rule and when do I use the addition rule?” Unfortunately, these questions have no easy answers.

You should construct a model that would allow you to continue counting the objects one by one if you had enough time.

If you can imagine the elements to be counted as being obtained through a multistep process then you can use the multiplication rule.

Some Advice about Counting

The total number of elements will be the product of the number of ways to perform each step. If, however, you can imagine the set of elements to be counted as being broken up into disjoint subsets, then you can use the addition rule.

The total number of elements in the set will be the sum of the number of elements in each subset.

One of the most common mistakes students make is to count certain possibilities more than once.

Example 11 – *Double Counting*

Consider again the problem of Example 7(**b**). A group consists of five men and seven women. How many teams of five contain at least one man?

Incorrect Solution

Imagine constructing the team as a two-step process:

Step 1: Choose a subset of one man from the five men.

Step 2: Choose a subset of four others from the remaining eleven people.

Example 11 – *Double Counting*

cont'd

Hence, by the multiplication rule, there are $\binom{5}{1} \cdot \binom{11}{4} = 1{,}650$ five-person teams that contain at least one man.

Analysis of the Incorrect Solution

The problem with the solution is that some teams are counted more than once. Suppose the men are Anwar, Ben, Carlos, Dwayne, and Ed and the women are Fumiko, Gail, Hui-Fan, Inez, Jill, Kim, and Laura.

According to the method described previously, one possible outcome of the two-step process is as follows:

Outcome of step 1: Anwar

Outcome of step 2: Ben, Gail, Inez, and Jill.

Example 11 – *Solution*

cont'd

In this case the team would be {Anwar, Ben, Gail, Inez, Jill}. But another possible outcome is

Outcome of step 1: Ben

Outcome of step 2: Anwar, Gail, Inez, and Jill,

which also gives the team {Anwar, Ben, Gail, Inez, Jill}.

Thus this one team is given by two different branches of the possibility tree, and so it is counted twice.

The Number of Partitions of a Set into *r* Subsets

The Number of Partitions of a Set into *r* Subsets

In an ordinary (or *singly indexed*) sequence, integers n are associated to numbers a_n. In a *doubly indexed* sequence, ordered pairs of integers (m, n) are associated to numbers $a_{m,n}$.

For example, combinations can be thought of as terms of the doubly indexed sequence defined by $C_{n,r} = \binom{n}{r}$ for all integers n and r with $0 \leq r \leq n$.

An important example of a doubly indexed sequence is the sequence of *Stirling numbers of the second kind*.

The Number of Partitions of a Set into r Subsets

Observe that if a set of three elements $\{x_1, x_2, x_3\}$ is partitioned into two subsets, then one of the subsets has one element and the other has two elements. Therefore, there are three ways the set can be partitioned:

$\{x_1, x_2\}\{x_3\}$	put x_3 by itself
$\{x_1, x_3\}\{x_2\}$	put x_2 by itself
$\{x_2, x_3\}\{x_1\}$	put x_1 by itself

In general, let

$S_{n,r}$ = number of ways a set of size n can be partitioned into r subsets

The Number of Partitions of a Set into *r* Subsets

Then, by the above, $S_{3.2} = 3$. The numbers $S_{n,r}$ are called **Stirling numbers of the second kind**.

Example 12 – *Values of Stirling Numbers*

Find $S_{4,1}$, $S_{4,2}$, $S_{4,3}$, and $S_{4,4}$.

Solution:

Given a set with four elements, denote it by $\{x_1, x_2, x_3, x_4\}$. The Stirling number $S_{4,1} = 1$ because a set of four elements can be partitioned into one subset in only one way:

$$\{x_1, x_2, x_3, x_4\}.$$

Similarly, $S_{4,4} = 1$ because there is only one way to partition a set of four elements into four subsets:

$$\{x_1\}\{x_2\}\{x_3\}\{x_4\}.$$

Example 12 – *Solution*

cont'd

The number $S_{4,2} = 7$. The reason is that any partition of $\{x_1, x_2, x_3, x_4\}$ into two subsets must consist either of two subsets of size two or of one subset of size three and one subset of size one.

The partitions for which both subsets have size two must pair x_1 with x_2, with x_3, or with x_4, which give rise to these three partitions:

$\{x_1, x_2\}\{x_3, x_4\}$ x_2 paired with x_1

$\{x_1, x_3\}\{x_2, x_4\}$ x_3 paired with x_1

$\{x_1, x_4\}\{x_2, x_3\}$ x_4 paired with x_1

Example 12 – *Solution*

cont'd

The partitions for which one subset has size one and the other has size three can have any one of the four elements in the subset of size one, which leads to these four partitions:

$\{x_1\}\{x_2, x_3, x_4\}$ x_1 by itself

$\{x_2\}\{x_1, x_3, x_4\}$ x_2 by itself

$\{x_3\}\{x_1, x_2, x_4\}$ x_3 by itself

$\{x_4\}\{x_1, x_2, x_3\}$ x_4 by itself

It follows that the total number of ways that the set $\{x_1, x_2, x_3, x_4\}$ can be partitioned into two subsets is $3 + 4 = 7$.

Example 12 – *Solution*

cont'd

Finally, $S_{4,3} = 6$ because any partition of a set of four elements into three subsets must have two elements in one subset and the other two elements in subsets by themselves.

There are $\binom{4}{2} = 6$ ways to choose the two elements to put together, which results in the following six possible partitions:

$$\{x_1, x_2\}\{x_3\}\{x_4\} \quad \{x_2, x_3\}\{x_1\}\{x_4\}$$

$$\{x_1, x_3\}\{x_2\}\{x_4\} \quad \{x_2, x_4\}\{x_1\}\{x_3\}$$

$$\{x_1, x_4\}\{x_2\}\{x_3\} \quad \{x_3, x_4\}\{x_1\}\{x_2\}$$