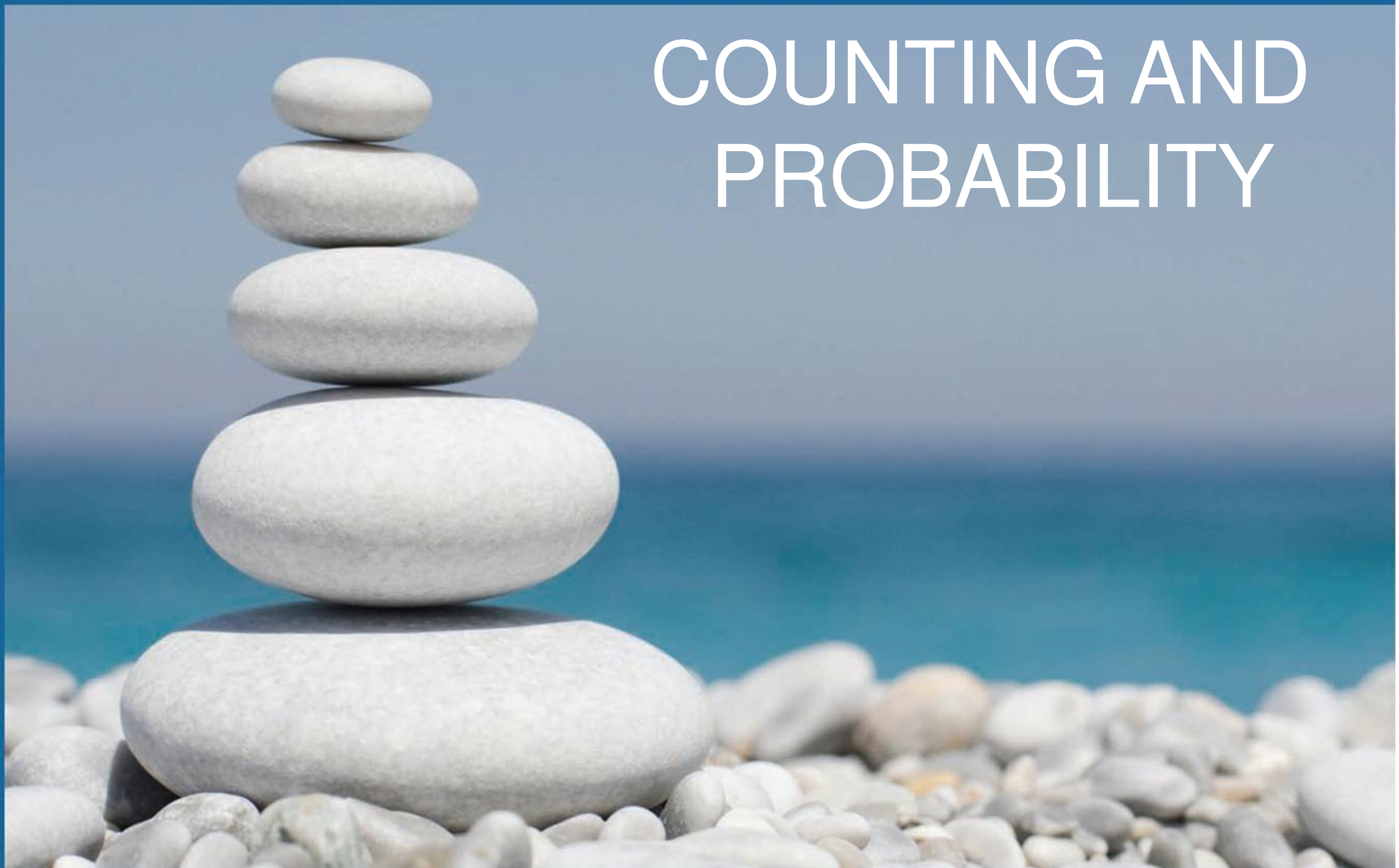
CHAPTER 9

COUNTING AND PROBABILITY



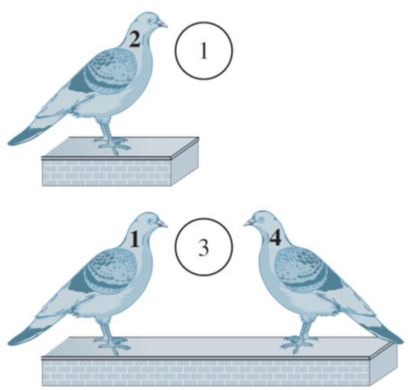
SECTION 9.4

The Pigeonhole Principle

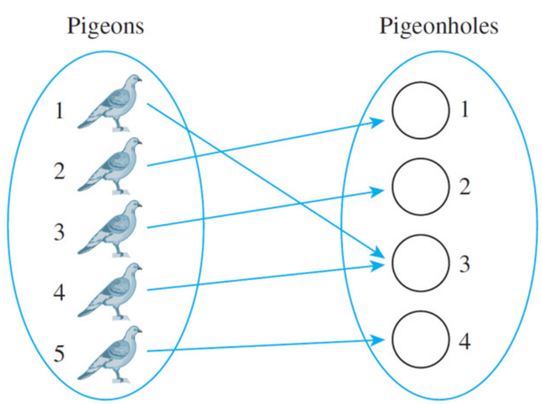
The Pigeonhole Principle

The pigeonhole principle states that if n pigeons fly into m pigeonholes and $n > m$, then at least one hole must contain two or more pigeons.

This principle is illustrated in Figure 9.4.1 for $n = 5$ and $m = 4$.



(a)



(b)

Figure 9.4.1

The Pigeonhole Principle

Illustration (a) shows the pigeons perched next to their holes, and (b) shows the correspondence from pigeons to pigeonholes.

The pigeonhole principle is sometimes called the *Dirichlet box principle* because it was first stated formally by J. P. G. L. Dirichlet (1805–1859).

Illustration (b) suggests the following mathematical way to phrase the principle.

Pigeonhole Principle

A function from one finite set to a smaller finite set cannot be one-to-one: There must be a least two elements in the domain that have the same image in the co-domain.

Example 1 – *Applying the Pigeonhole Principle*

a. In a group of six people, must there be at least two who were born in the same month?

In a group of thirteen people, must there be at least two who were born in the same month? Why?

b. Among the residents of New York City, must there be at least two people with the same number of hairs on their heads? Why?

Example 1(a) – *Solution*

A group of six people need not contain two who were born in the same month. For instance, the six people could have birthdays in each of the six months January through June.

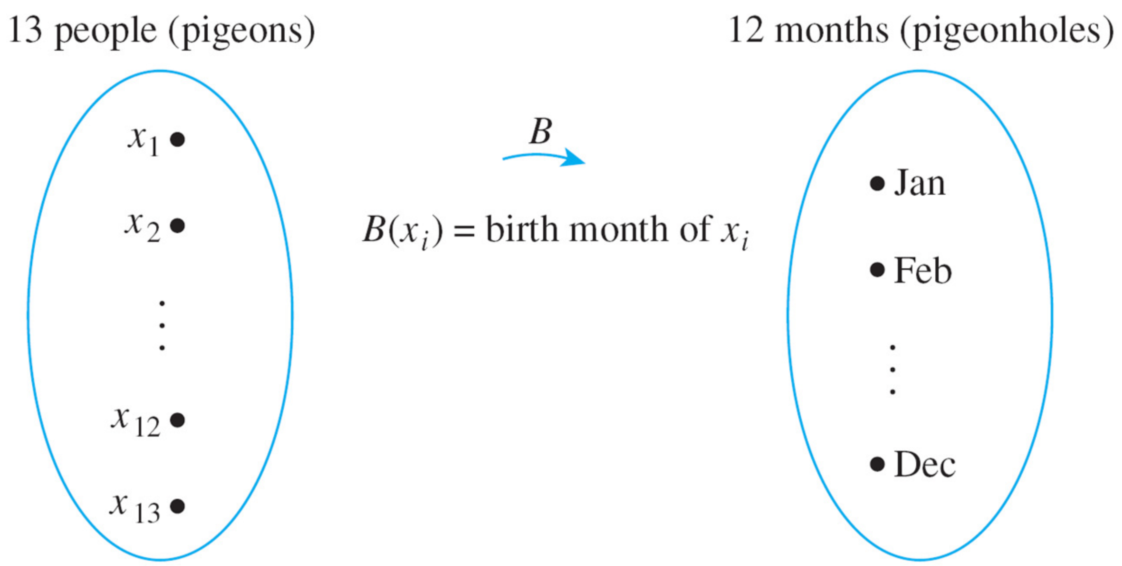
A group of thirteen people, however, must contain at least two who were born in the same month, for there are only twelve months in a year and $13 > 12$.

To get at the essence of this reasoning, think of the thirteen people as the pigeons and the twelve months of the year as the pigeonholes.

Example 1(a) – *Solution*

cont'd

Denote the thirteen people by the symbols $x_1, x_2, \ldots, x_{13}$ and define a function B from the set of people to the set of twelve months as shown in the following arrow diagram.



Example 1(a) – *Solution*

cont'd

The pigeonhole principle says that no matter what the particular assignment of months to people, there must be at least two arrows pointing to the same month.

Thus at least two people must have been born in the same month.

Example 1(b) – *Solution*

cont'd

The answer is yes.

In this example the pigeons are the people of New York City and the pigeonholes are all possible numbers of hairs on any individual's head.

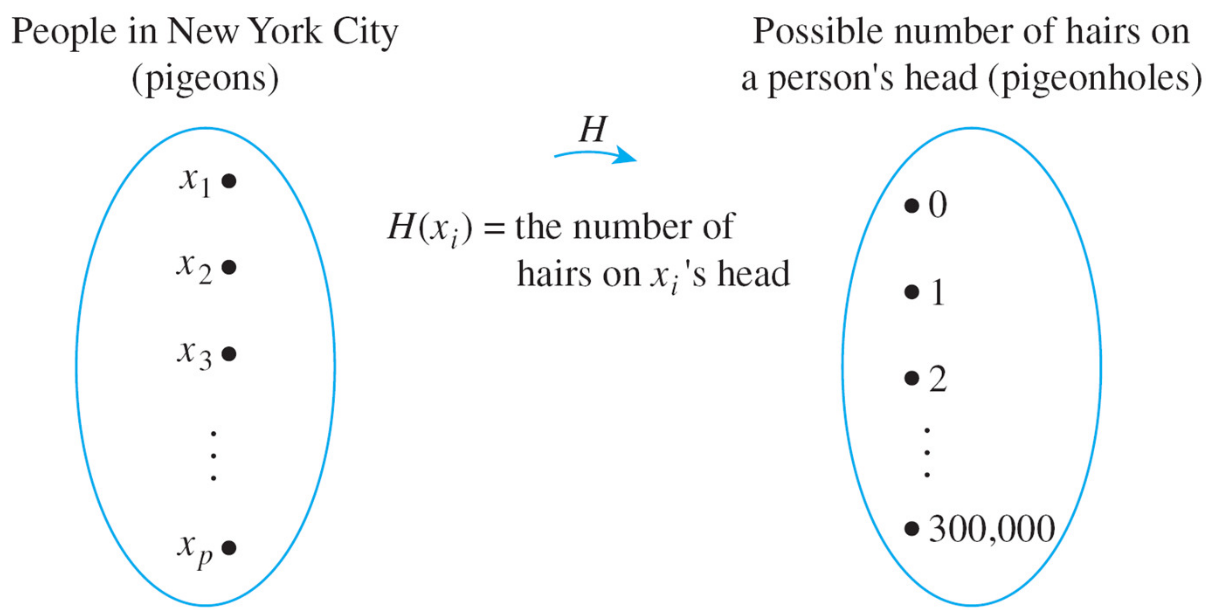
Call the population of New York City P. It is known that P is at least 5,000,000.

Also the maximum number of hairs on any person's head is known to be no more than 300,000.

Example 1(b) – *Solution*

cont'd

Define a function *H* from the set of people in New York City $\{x_1, x_2, \ldots, x_p\}$ to the set {0, 1, 2, 3, . . . , 300 000}, as shown below.



Example 1(b) – *Solution*

cont'd

Since the number of people in New York City is larger than the number of possible hairs on their heads, the function *H* is not one-to-one; at least two arrows point to the same number.

But that means that at least two people have the same number of hairs on their heads.

Application to Decimal Expansions of Fractions

Application to Decimal Expansions of Fractions

One important consequence of the piegonhole principle is the fact that

the decimal expansion of any rational number either terminates or repeats.

A terminating decimal is one like

$$3.625,$$

and a repeating decimal is one like

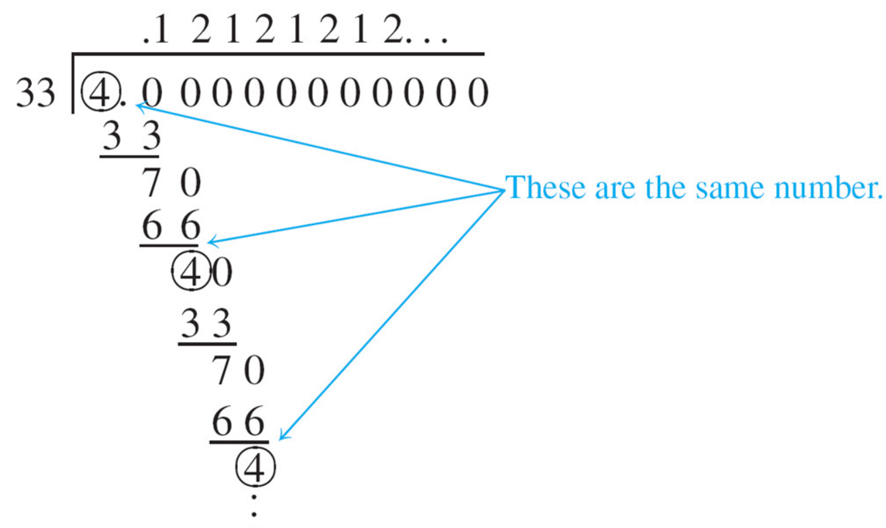
$$2.38\overline{246},$$

where the bar over the digits 246 means that these digits are repeated forever.

Application to Decimal Expansions of Fractions

A rational number is one that can be written as a ratio of integers—in other words, as a fraction.

The decimal expansion of a fraction is obtained by dividing its numerator by its denominator using long division. For example, the decimal expansion of 4/33 is obtained as follows:



Application to Decimal Expansions of Fractions

Because the number 4 reappears as a remainder in the long-division process, the sequence of quotients and remainders that give the digits of the decimal expansion repeats forever; hence the digits of the decimal expansion repeat forever.

In general, when one integer is divided by another, it is the pigeonhole principle (together with the quotient-remainder theorem) that guarantees that such a repetition of remainders and hence decimal digits must always occur.

This is explained in the next example.

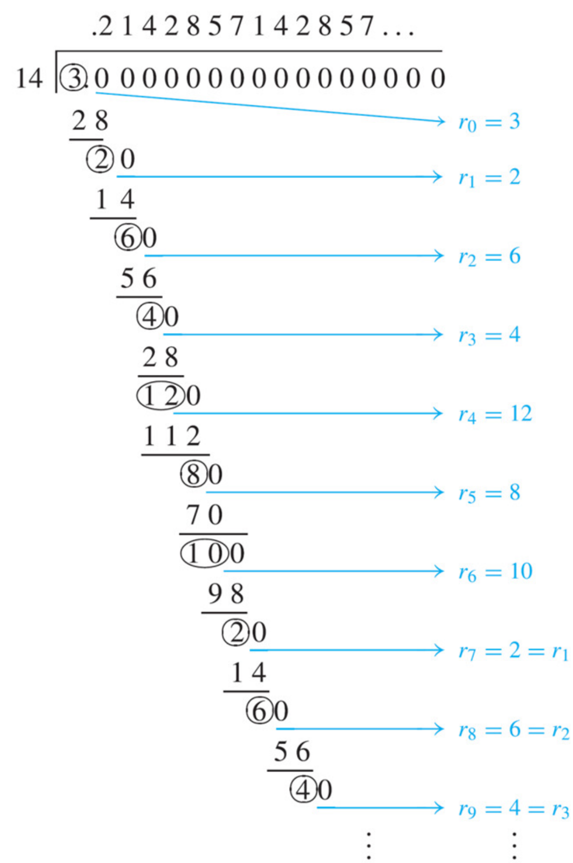
Application to Decimal Expansions of Fractions

The analysis in the example uses an obvious generalization of the pigeonhole principle, namely that a function from an infinite set to a finite set cannot be one-to-one.

Example 4 – *The Decimal Expansion of a Fraction*

Consider a fraction *a*/*b*, where for simplicity *a* and *b* are both assumed to be positive.

The decimal expansion of *a*/*b* is obtained by dividing the *a* by the *b* as illustrated here for *a* = 3 and *b* = 14.



Example 4 – *The Decimal Expansion of a Fraction* cont'd

Let $r_0 = a$ and let $r_1, r_2, r_3, \ldots$ be the successive remainders obtained in the long division of a by b.

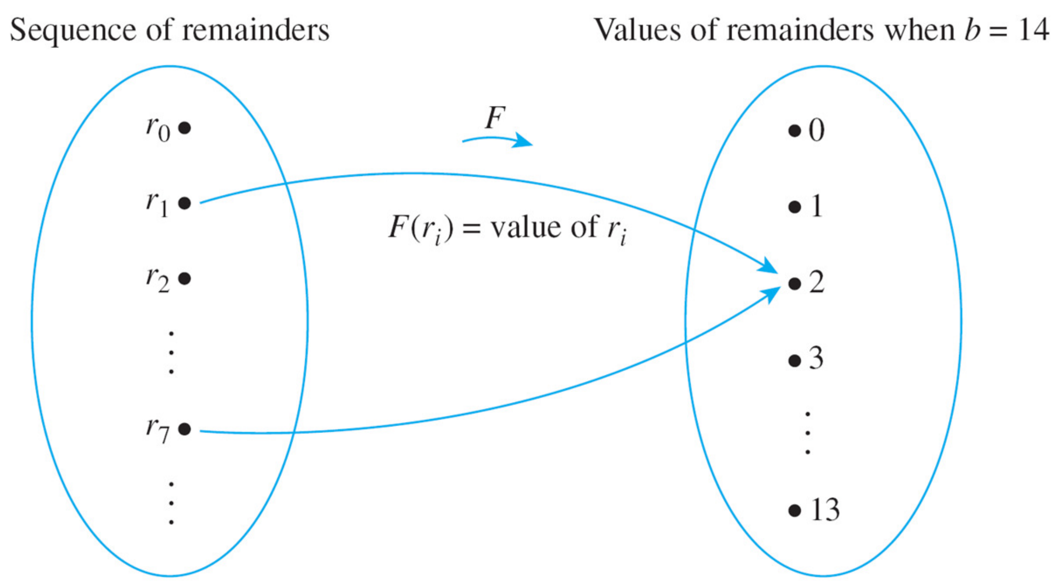
By the quotient-remainder theorem, each remainder must be between 0 and $b - 1$. (In this example, a is 3 and b is 14, and so the remainders are from 0 to 13.)

If some remainder $r_i = 0$, then the division terminates and a/b has a terminating decimal expansion. If no $r_i = 0$, then the division process and hence the sequence of remainders continues forever.

Example 4 – *The Decimal Expansion of a Fraction* cont'd

By the pigeonhole principle, since there are more remainders than values that the remainders can take, some remainder value must repeat: $r_j = r_k$, for some indices j and k with $j < k$.

This is illustrated below for $a = 3$ and $b = 14$.



Example 4 – *The Decimal Expansion of a Fraction* cont'd

If follows that the decimal digits obtained from the divisions between r_j and r_{k-1} repeat forever.

In the case of 3/14, the repetition begins with $r_7 = 2 = r_1$ and the decimal expansion repeats the quotients obtained from the divisions from r_1 through r_6 forever:

$$3/14 = 0.2\overline{142857}.$$

Generalized Pigeonhole Principle

Generalized Pigeonhole Principle

A generalization of the pigeonhole principle states that if n pigeons fly into m pigeonholes and, for some positive integer k, $k < n/m$, then at least one pigeonhole contains $k + 1$ or more pigeons.

Generalized Pigeonhole Principle

For any function f from a finite set X with n elements to a finite set Y with m elements and for any positive integer k, if $k < n/m$, then there is some $y \in Y$ such that y is the image of at least $k + 1$ distinct elements of X.

Example 5 – *Applying the Generalized Pigeonhole Principle*

Show how the generalized pigeonhole principle implies that in a group of 85 people, at least 4 must have the same last initial.

Solution:

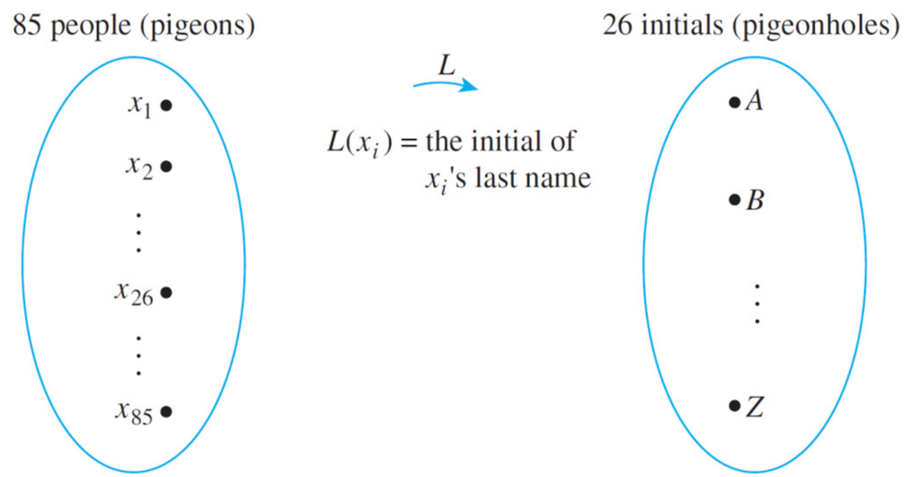
In this example the pigeons are the 85 people and the pigeonholes are the 26 possible last initials of their names. Note that

$$3 < 85/26 \cong 3.27.$$

Example 5 – *Solution*

cont'd

Consider the function *L* from people to initials defined by the following arrow diagram.



Since $3 < 85/26$, the generalized pigeonhole principle states that some initial must be the image of at least four $(3 + 1)$ people.

Thus at least four people have the same last initial.

Generalized Pigeonhole Principle

Consider the following contrapositive form of the generalized pigeonhole principle.

Generalized Pigenohole Principle (Contrapositive Form)

For any function f from a finite set X with n elements to a finite set Y with m elements and for any positive integer k, if for each $y \in Y$, $f^{-1}(y)$ has at most k elements, then X has at most km elements; in other words, $n \leq km$.

You may find it natural to use the contrapositive form of the generalized pigeonhole principle in certain situations.

Generalized Pigeonhole Principle

For instance, the result of Example 5 can be explained as follows:

Suppose no 4 people out of the 85 had the same last initial. Then at most 3 would share any particular one.

By the generalized pigeonhole principle (contrapositive form), this would imply that the total number of people is at most $3 \cdot 26 = 78$. But this contradicts the fact that there are 85 people in all.

Hence at least 4 people share a last initial.

Example 6 – *Using the Contrapositive Form of the Generalized Pigeonhole Principle*

There are 42 students who are to share 12 computers. Each student uses exactly 1 computer, and no computer is used by more than 6 students. Show that at least 5 computers are used by 3 or more students.

Example 6(a) – *Solution*

cont'd

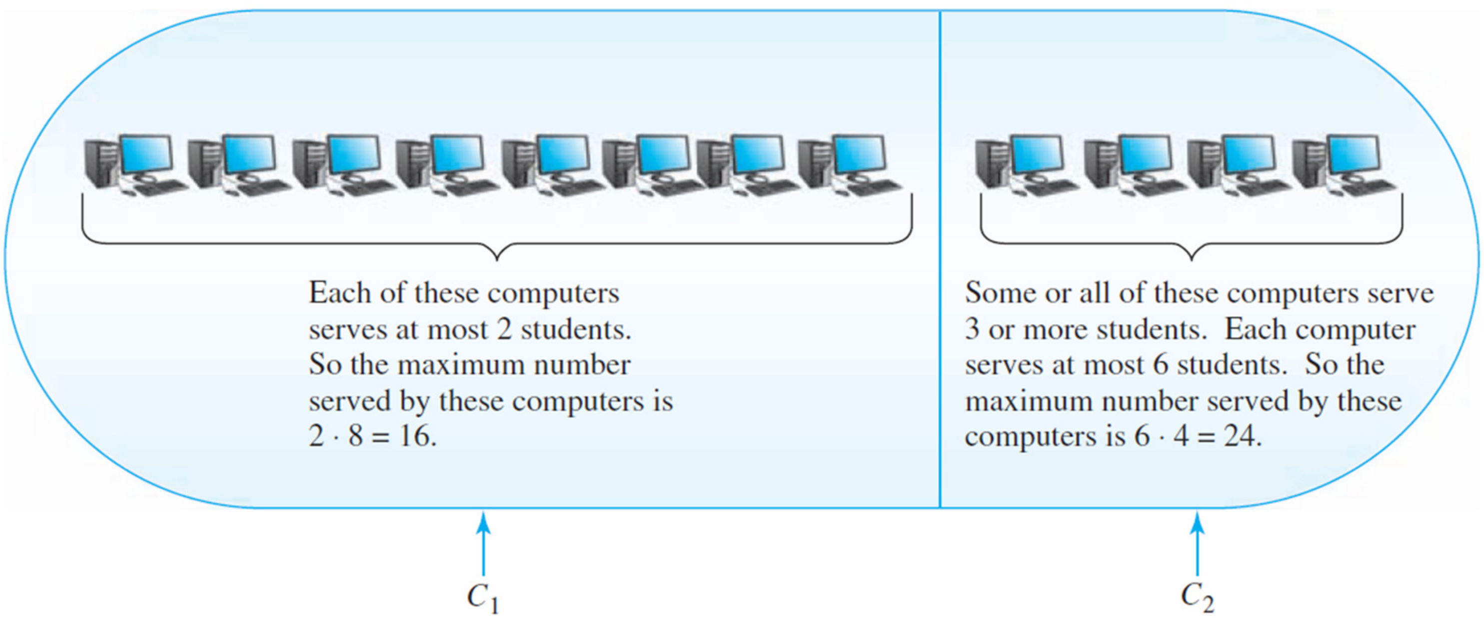
Using an Argument by Contradiction: Suppose not. Suppose that 4 or fewer computers are used by 3 or more students. *[A contradiction will be derived.]* Then 8 or more computers are used by 2 or fewer students.

Divide the set of computers into two subsets: C_1 and C_2.

Example 6(a) – *Solution*

cont'd

Into C_1 place 8 of the computers used by 2 or fewer students; into C_2 place the computers used by 3 or more students plus any remaining computers (to make a total of 4 computers in C_2). (See Figure 9.4.3.)



The set of 12 computers

Figure 9.4.3

Example 6(a) – *Solution*

cont'd

Since at most 6 students are served by any one computer, by the contrapositive form of the generalized pigeonhole principle, the computers in set C_2 serve at most
$6 \cdot 4 = 24$ students.

Since at most 2 students are served by any one computer in C_1, by the generalized pigeonhole principle (contrapositive form), the computers in set C_1 serve at most
$2 \cdot 8 = 16$ students.

Example 6(a) – *Solution*

cont'd

Hence the total number of students served by the computers is 24 + 16 = 40.

But this contradicts the fact that each of the 42 students is served by a computer.

Therefore, the supposition is false: At least 5 computers are used by 3 or more students.

Example 6(b) – *Solution*

cont'd

Using a Direct Argument: Let k be the number of computers used by 3 or more students. *[We must show that $k \geq 5$.]*

Because each computer is used by at most 6 students, these computers are used by at most $6k$ students (by the contrapositive form of the generalized pigeonhole principle).

Each of the remaining $12 - k$ computers is used by at most 2 students.

Example 6(b) – *Solution*

cont'd

Hence, taken together, they are used by at most $2(12 - k) = 24 - 2k$ students (again, by the contrapositive form of the generalized pigeonhole principle).

Thus the maximum number of students served by the computers is $6k + (24 - 2k) = 4k + 24$.

Because 42 students are served by the computers, $4k + 24 \geq 42$.

Solving for k gives that $k \geq 4.5$, and since k is an integer, this implies that $k \geq 5$ *[as was to be shown]*.

Proof of the Pigeonhole Principle

Proof of the Pigeonhole Principle

The truth of the pigeonhole principle depends essentially on the sets involved being finite.

We have known that a set is called **finite** if, and only if, it is the empty set or there is a one-to-one correspondence from $\{1, 2, . . . , n\}$ to it, where n is a positive integer.

In the first case the **number of elements** in the set is said to be 0, and in the second case it is said to be n. A set that is not finite is called **infinite.**

Thus any finite set is either empty or can be written in the form $\{x_1, x_2, . . . , x_n\}$ where n is a positive integer.

Proof of the Pigeonhole Principle

Theorem 9.4.1 The Pigeonhole Principle

For any function f from a finite set X with n elements to a finite set Y with m elements, if $n > m$, then f is not one-to-one.

Proof:

Suppose f is any function from a finite set X with n elements to a finite set Y with m elements where $n > m$. Denote the elements of Y by $y_1, y_2, \ldots, y_m$.

We have known that for each y_i in Y, the inverse image set $f^{-1}(y_i) = \{x \in X \mid f(x) = y_i\}$.

Proof of the Pigeonhole Principle

Now consider the collection of all the inverse image sets for all the elements of *Y*:

$$f^{-1}(y_1),\ f^{-1}(y_2),\ \ldots,\ f^{-1}(y_m).$$

By definition of function, each element of *X* is sent by *f* to some element of *Y*. Hence each element of *X* is in one of the inverse image sets, and so the union of all these sets equals *X*.

But also, by definition of function, no element of *X* is sent by *f* to more than one element of *Y*.

Thus each element of *X* is in only one of the inverse image sets, and so the inverse image sets are mutually disjoint.

Proof of the Pigeonhole Principle

By the addition rule, therefore,

$$N(X) = N(f^{-1}(y_1)) + N(f^{-1}(y_2)) + \cdots + N(f^{-1}(y_m)). \qquad 9.4.1$$

Now suppose that *f is* one-to-one *[which is the opposite of what we want to prove]*.

Then each set $f^{-1}(y_i)$ has at most one element, and so

$$N(f^{-1}(y_1)) + N(f^{-1}(y_2)) + \cdots + N(f^{-1}(y_m)) \leq \underbrace{1 + 1 + \cdots + 1}_{m \text{ terms}} = m \qquad 9.4.2$$

Proof of the Pigeonhole Principle

Putting equations (9.4.1) and (9.4.2) together gives that

$$n = N(X) \leq m = N(Y).$$

This contradicts the fact that $n > m$, and so the supposition that f is one-to-one must be false.

Hence f is not one-to-one *[as was to be shown]*.

Proof of the Pigeonhole Principle

An important theorem that follows from the pigeonhole principle states that a function from one finite set to another finite set of the same size is one-to-one if, and only if, it is onto.

As we have known, this result does not hold for infinite sets.

Proof of the Pigeonhole Principle

Note that Theorem 9.4.2 applies in particular to the case $X = Y$.

Theorem 9.4.2 One-to-One and Onto for Finite Sets

Let X and Y be finite sets with the same number of elements and suppose f is a function from X to Y. Then f is one-to-one if, and only if, f is onto.

Thus a one-to-one function from a finite set to itself is onto, and an onto function from a finite set to itself is one-to-one.

Such functions are permutations of the sets on which they are defined.

Proof of the Pigeonhole Principle

For instance, the function defined by the diagram below, is another representation for the permutation *cdba* obtained by listing the images of *a*, *b*, *c*, and *d* in order.