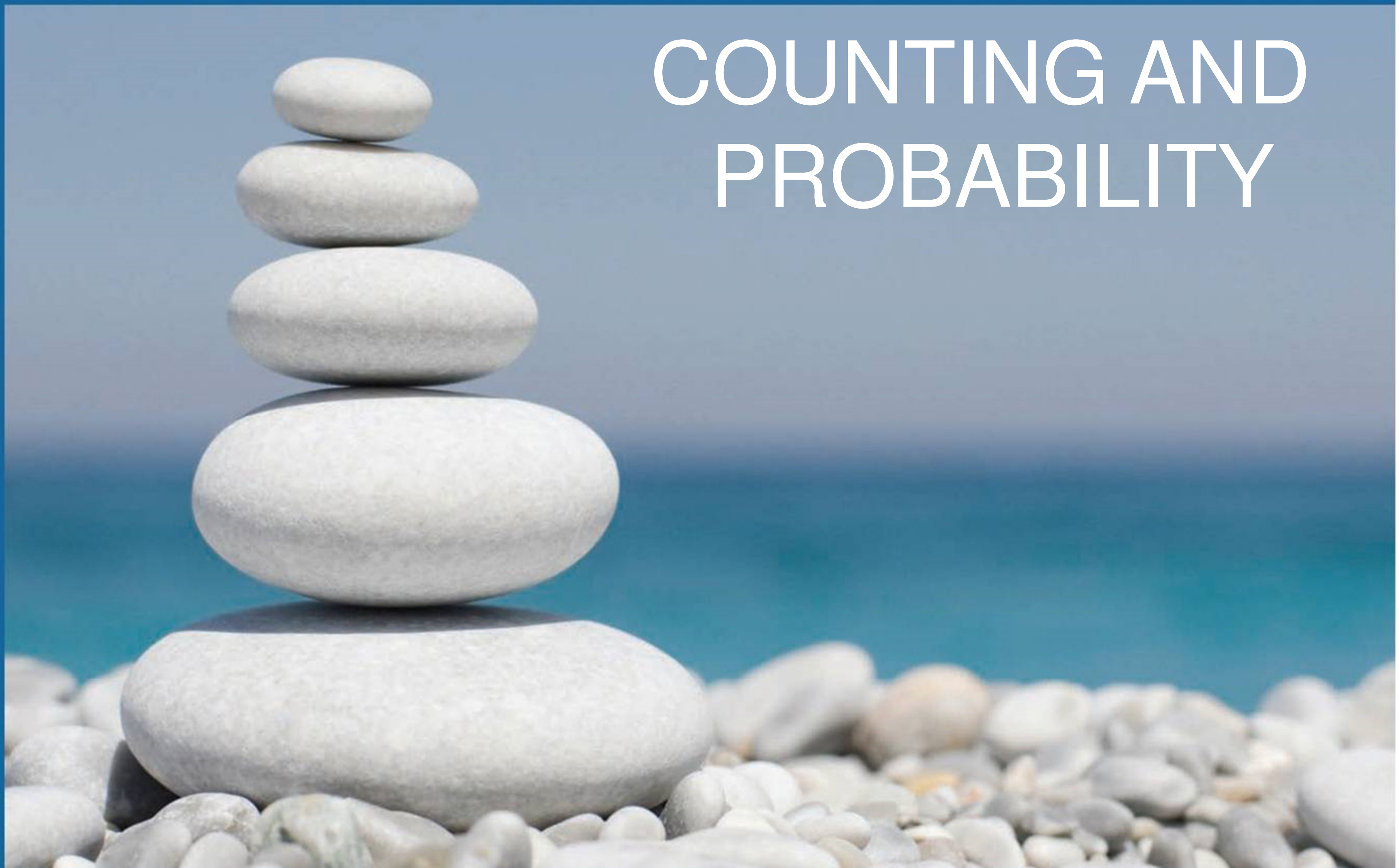
CHAPTER 9

# COUNTING AND PROBABILITY



SECTION 9.2

# Possibility Trees and the Multiplication Rule

# Possibility Trees and the Multiplication Rule

A tree structure is a useful tool for keeping systematic track of all possibilities in situations in which events happen in order.

The next example shows how to use such a structure to count the number of different outcomes of a tournament.

# Example 1 – *Possibilities for Tournament Play*

Teams *A* and *B* are to play each other repeatedly until one wins two games in a row or a total of three games.

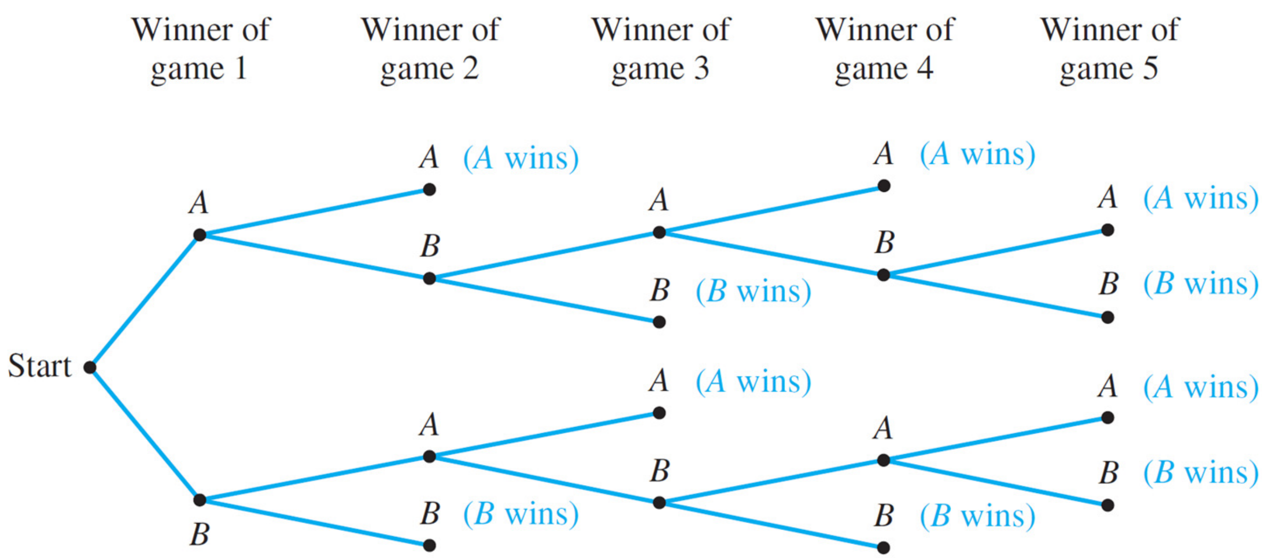
One way in which this tournament can be played is for *A* to win the first game, *B* to win the second, and *A* to win the third and fourth games. Denote this by writing *A*–*B*–*A*–*A*.

**a.** How many ways can the tournament be played?

**b.** Assuming that all the ways of playing the tournament are equally likely, what is the probability that five games are needed to determine the tournament winner?

# Example 1(a) – *Solution*

The possible ways for the tournament to be played are represented by the distinct paths from "root" (the start) to "leaf" (a terminal point) in the tree shown sideways in Figure 9.2.1.




The Outcomes of a Tournament

**Figure 9.2.1**

# Example 1(a) – *Solution*

cont'd

The label on each branching point indicates the winner of the game. The notations in parentheses indicate the winner of the tournament.

The fact that there are ten paths from the root of the tree to its leaves shows that there are ten possible ways for the tournament to be played.

They are (moving from the top down): *A–A*, *A–B–A–A*, *A–B–A–B–A*, *A–B–A–B–B*, *A–B–B*, *B–A–A*, *B–A–B–A–A*, *B–A–B–A–B*, *B–A–B–B*, and *B–B*.

In five cases *A* wins, and in the other five *B* wins. The least number of games that must be played to determine a winner is two, and the most that will need to be played is five.

# Example 1(b) – *Solution*

cont'd

Since all the possible ways of playing the tournament listed in part (a) are assumed to be equally likely, and the listing shows that five games are needed in four different cases (*A*–*B*–*A*–*B*–*A*, *A*–*B*–*A*–*B*–*B*, *B*–*A*–*B*–*A*–*B*, and *B*–*A*–*B*–*A*–*A*), the probability that five games are needed is

$$4/10 = 2/5$$

$$= 40\%.$$

# The Multiplication Rule

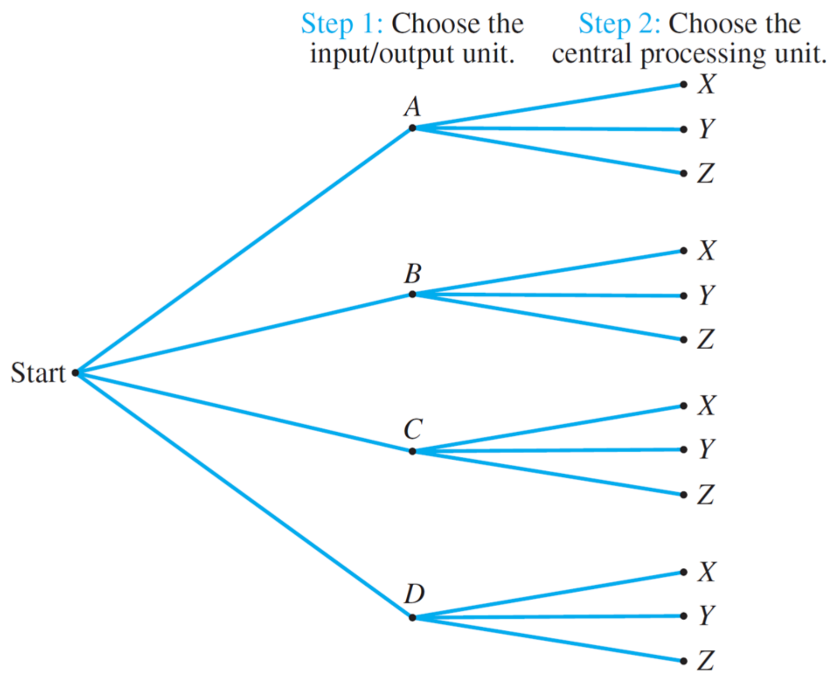
# The Multiplication Rule

Consider the following example. Suppose a computer installation has four input/output units (*A*, *B*, *C*, and *D*) and three central processing units (*X*, *Y*, and *Z*).

Any input/output unit can be paired with any central processing unit. How many ways are there to pair an input/output unit with a central processing unit?

# The Multiplication Rule

The possible outcomes of this operation are illustrated in the possibility tree of Figure 9.2.2.




Pairing Objects Using a Possibility Tree

**Figure 9.2.2**

# The Multiplication Rule

Thus the total number of ways to pair the two types of units is the same as the number of branches of the tree, which is

$$3 + 3 + 3 + 3 = 4 \cdot 3 = 12.$$

The idea behind this example can be used to prove the following rule.

**Theorem 9.2.1 The Multiplication Rule**

If an operation consists of $k$ steps and

the first step can be performed in $n_1$ ways,

the second step can be performed in $n_2$ ways *[regardless of how the first step was performed]*,

$\vdots$

the $k$th step can be performed in $n_k$ ways *[regardless of how the preceding steps were performed]*,

then the entire operation can be performed in $n_1 n_2 \cdots n_k$ ways.

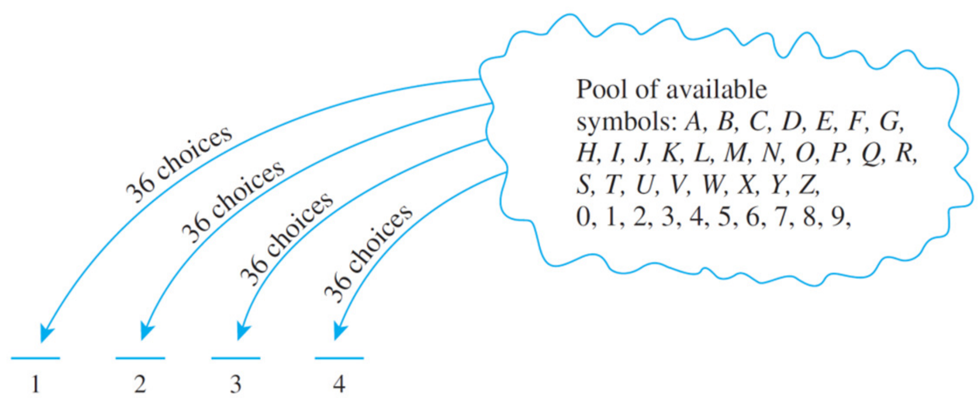
## Example 2 – *Number of Personal Identification Numbers (PINs)*

A typical PIN (personal identification number) is a sequence of any four symbols chosen from the 26 letters in the alphabet and the ten digits, with repetition allowed. How many different PINs are possible?

Solution:

Typical PINs are CARE, 3387, B32B, and so forth.

You can think of forming a PIN as a four-step operation to fill in each of the four symbols in sequence.



# Example 2 – *Solution*

cont'd

**Step 1:** Choose the first symbol.

**Step 2:** Choose the second symbol.

**Step 3:** Choose the third symbol.

**Step 4:** Choose the fourth symbol.

There is a fixed number of ways to perform each step, namely 36, regardless of how preceding steps were performed.

And so, by the multiplication rule, there are $36 \cdot 36 \cdot 36 \cdot 36 = 36^4 = 1,679,616$ PINs in all.

# Example 4 – *Number of PINs without Repetition*

In Example 2 we formed PINs using four symbols, either letters of the alphabet or digits, and supposing that letters could be repeated. Now suppose that repetition is not allowed.

**a.** How many different PINs are there?

**b.** If all PINs are equally likely, what is the probability that a PIN chosen at random contains no repeated symbol?

Solution:

**a.** Again think of forming a PIN as a four-step operation: Choose the first symbol, then the second, then the third, and then the fourth.

# Example 4 – *Solution*

cont'd

There are 36 ways to choose the first symbol, 35 ways to choose the second (since the first symbol cannot be used again), 34 ways to choose the third (since the first two symbols cannot be reused), and 33 ways to choose the fourth (since the first three symbols cannot be reused).

Thus, the multiplication rule can be applied to conclude that there are $36 \cdot 35 \cdot 34 \cdot 33 = 1{,}413{,}720$ different PINs with no repeated symbol.

**b.** By part (a) there are 1,413,720 PINs with no repeated symbol, and by Example 2 there are 1,679,616 PINs in all.

# Example 4 – *Solution*

cont'd

Thus the probability that a PIN chosen at random contains no repeated symbol is $\frac{1{,}413{,}720}{1{,}679{,}616} \cong .8417$. In other words, approximately 84% of PINs have no repeated symbol.

# The Multiplication Rule

We have known that if *S* is a nonempty, finite set of characters, then a string over *S* is a finite sequence of elements of *S*.

The number of characters in a string is called the **length** of the string. The **null string over *S*** is the “string” with no characters. It is usually denoted $\varepsilon$ and is said to have length 0.

## When the Multiplication Rule Is Difficult or Impossible to Apply

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Consider the following problem:

Three officers—a president, a treasurer, and a secretary—are to be chosen from among four people: Ann, Bob, Cyd, and Dan. Suppose that, for various reasons, Ann cannot be president and either Cyd or Dan must be secretary. How many ways can the officers be chosen?

It is natural to try to solve this problem using the multiplication rule. A person might answer as follows:

There are three choices for president (all except Ann), three choices for treasurer (all except the one chosen as president), and two choices for secretary (Cyd or Dan).

## When the Multiplication Rule Is Difficult or Impossible to Apply

Therefore, by the multiplication rule, there are $3 \cdot 3 \cdot 2 = 18$ choices in all.

Unfortunately, this analysis is incorrect. The number of ways to choose the secretary varies depending on who is chosen for president and treasurer.

For instance, if Bob is chosen for president and Ann for treasurer, then there are two choices for secretary: Cyd and Dan.

But if Bob is chosen for president and Cyd for treasurer, then there is just one choice for secretary: Dan.

# When the Multiplication Rule Is Difficult or Impossible to Apply

The clearest way to see all the possible choices is to construct the possibility tree, as is shown in Figure 9.2.3.

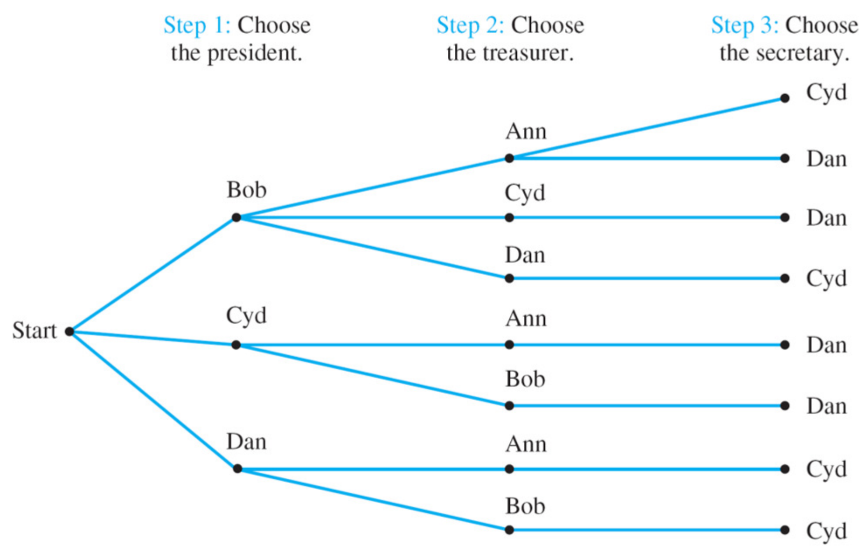


**Figure 9.2.3**

From the tree it is easy to see that there are only eight ways to choose a president, treasurer, and secretary so as to satisfy the given conditions.

## Example 7 – *A More Subtle Use of the Multiplication Rule*

Reorder the steps for choosing the officers in the previous example so that the total number of ways to choose officers can be computed using the multiplication rule.

Solution:

**Step 1:** Choose the secretary.

**Step 2:** Choose the president.

**Step 3:** Choose the treasurer.

# Example 7 – *Solution*

cont'd

There are exactly two ways to perform step 1 (either Cyd or Dan may be chosen), two ways to perform step 2 (neither Ann nor the person chosen in step 1 may be chosen but either of the other two may), and two ways to perform step 3 (either of the two people not chosen as secretary or president may be chosen as treasurer).

Thus, by the multiplication rule, the total number of ways to choose officers is $2 \cdot 2 \cdot 2 = 8$.

# Example 7 – *Solution*

cont'd

A possibility tree illustrating this sequence of choices is shown in Figure 9.2.4.

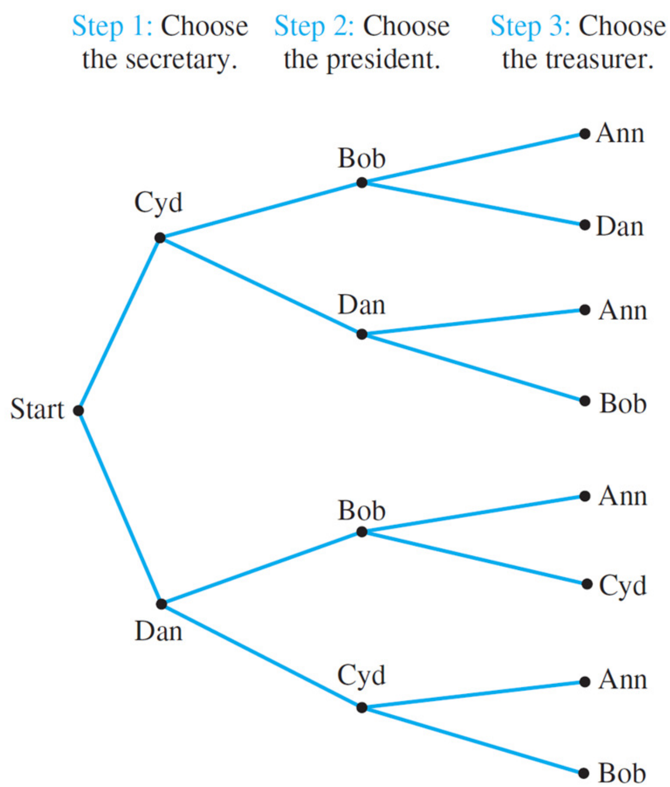


**Figure 9.2.4**

# Example 7 – *Solution*

cont'd

Note how balanced the earlier tree in Figure 9.2.4 is compared with the one in Figure 9.2.3.

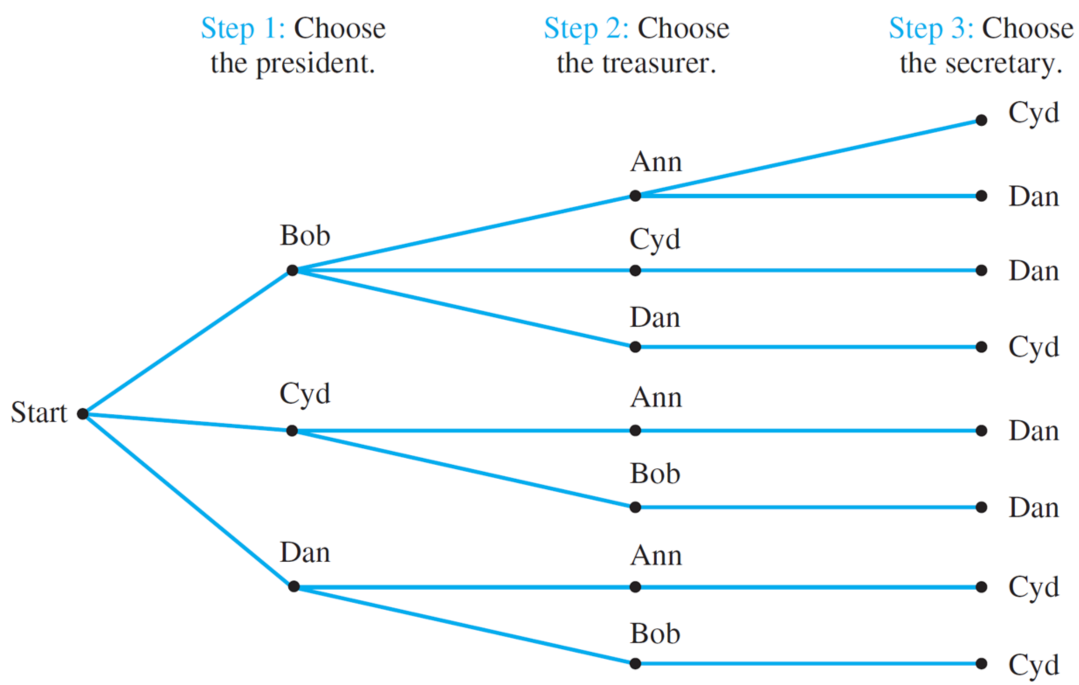


**Figure 9.2.3**

# Permutations

# Permutations

A **permutation** of a set of objects is an ordering of the objects in a row. For example, the set of elements *a*, *b*, and *c* has six permutations.

*abc* *acb* *cba* *bac* *bca* *cab*

In general, given a set of *n* objects, how many permutations does the set have? Imagine forming a permutation as an *n*-step operation:

**Step 1:** Choose an element to write first.

# Permutations

**Step 2:** Choose an element to write second.

⋮ ⋮

**Step *n*:** Choose an element to write *n*th.

Any element of the set can be chosen in step 1, so there are *n* ways to perform step 1.

Any element except that chosen in step 1 can be chosen in step 2, so there are $n - 1$ ways to perform step 2.

# Permutations

In general, the number of ways to perform each successive step is one less than the number of ways to perform the preceding step.

At the point when the $n$th element is chosen, there is only one element left, so there is only one way to perform step $n$.

Hence, by the multiplication rule, there are

$$n(n-1)(n-2) \cdots 2 \cdot 1 = n!$$

ways to perform the entire operation.

# Permutations

In other words, there are $n!$ permutations of a set of $n$ elements. This reasoning is summarized in the following theorem.

**Theorem 9.2.2**

For any integer $n$ with $n \geq 1$, the number of permutations of a set with $n$ elements is $n!$.

# Example 8 – *Permutations of the Letters in a Word*

**a.** How many ways can the letters in the word *COMPUTER* be arranged in a row?

**b.** How many ways can the letters in the word *COMPUTER* be arranged if the letters *CO* must remain next to each other (in order) as a unit?

**c.** If letters of the word *COMPUTER* are randomly arranged in a row, what is the probability that the letters *CO* remain next to each other (in order) as a unit?

# Example 8 – *Solution*

**a.** All the eight letters in the word *COMPUTER* are distinct, so the number of ways in which we can arrange the letters equals the number of permutations of a set of eight elements. This equals 8! = 40,320.

**b.** If the letter group *CO* is treated as a unit, then there are effectively only seven objects that are to be arranged in a row.

| CO | M | P | U | T | E | R |
|---|---|---|---|---|---|---|

Hence there are as many ways to write the letters as there are permutations of a set of seven elements, namely 7! = 5,040.

# Example 8 – *Solution*

cont'd

**c.** When the letters are arranged randomly in a row, the total number of arrangements is 40,320 by part (a), and the number of arrangements with the letters *CO* next to each other (in order) as a unit is 5,040.

Thus the probability is

$$\frac{5{,}040}{40{,}320} = \frac{1}{8}$$

$$= 12.5\%.$$

# Permutations of Selected Elements

# Permutations of Selected Elements

Given the set {*a*, *b*, *c*}, there are six ways to select two letters from the set and write them in order.

*ab*  *ac*  *ba*  *bc*  *ca*  *cb*

Each such ordering of two elements of {*a*, *b*, *c*} is called a 2-*permutation* of {*a*, *b*, *c*}.

**• Definition**

An ***r*-permutation** of a set of $n$ elements is an ordered selection of $r$ elements taken from the set of $n$ elements. The number of $r$-permutations of a set of $n$ elements is denoted $\boldsymbol{P(n, r)}$.

# Permutations of Selected Elements

**Theorem 9.2.3**

If $n$ and $r$ are integers and $1 \leq r \leq n$, then the number of $r$-permutations of a set of $n$ elements is given by the formula

$$P(n, r) = n(n-1)(n-2)\cdots(n-r+1) \qquad \text{first version}$$

or, equivalently,

$$P(n, r) = \frac{n!}{(n-r)!} \qquad \text{second version.}$$

# Example 10 – *Evaluating r-Permutations*

**a.** Evaluate $P(5, 2)$.

**b.** How many 4-permutations are there of a set of seven objects?

**c.** How many 5-permutations are there of a set of five objects?

Solution:

**a.** $$P(5, 2) = \frac{5!}{(5-2)!} = \frac{5 \cdot 4 \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}}{\cancel{3} \cdot \cancel{2} \cdot \cancel{1}}$$

$$= 20$$

# Example 10 – *Solution*

cont'd

**b.** The number of 4-permutations of a set of seven objects is

$$P(7, 4) = \frac{7!}{(7-4)!}$$

$$= \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}}{\cancel{3} \cdot \cancel{2} \cdot \cancel{1}}$$

$$= 7 \cdot 6 \cdot 5 \cdot 4$$

$$= 840.$$

# Example 10 – *Solution*

cont'd

**c.** The number of 5-permutations of a set of five objects is

$$\begin{aligned} P(5,5) &= \frac{5!}{(5-5)!} \\ &= \frac{5!}{0!} \\ &= \frac{5!}{1} \\ &= 5! = 120. \end{aligned}$$

Note that the definition of 0! as 1 makes this calculation come out as it should, for the number of 5-permutations of a set of five objects is certainly equal to the number of permutations of the set.

# Example 12 – *Proving a Property of P (n, r)*

Prove that for all integers $n \geq 2$,

$$P(n, 2) + P(n, 1) = n^2.$$

Solution:

Suppose $n$ is an integer that is greater than or equal to 2.

By Theorem 9.2.3,

$$P(n, 2) = \frac{n!}{(n-2)!}$$

$$= \frac{n(n-1)\cancel{(n-2)!}}{\cancel{(n-2)!}}$$

# Example 12 – *Solution*

cont'd

$$= n(n - 1)$$

and

$$P(n, 1) = \frac{n!}{(n-1)!}$$

$$= \frac{n \cdot \cancel{(n-1)!}}{\cancel{(n-1)!}} = n.$$

Hence

$$P(n, 2) + P(n, 1) = n \cdot (n - 1) + n$$

$$= n^2 - n + n$$

$$= n^2,$$

which is what we needed to show.