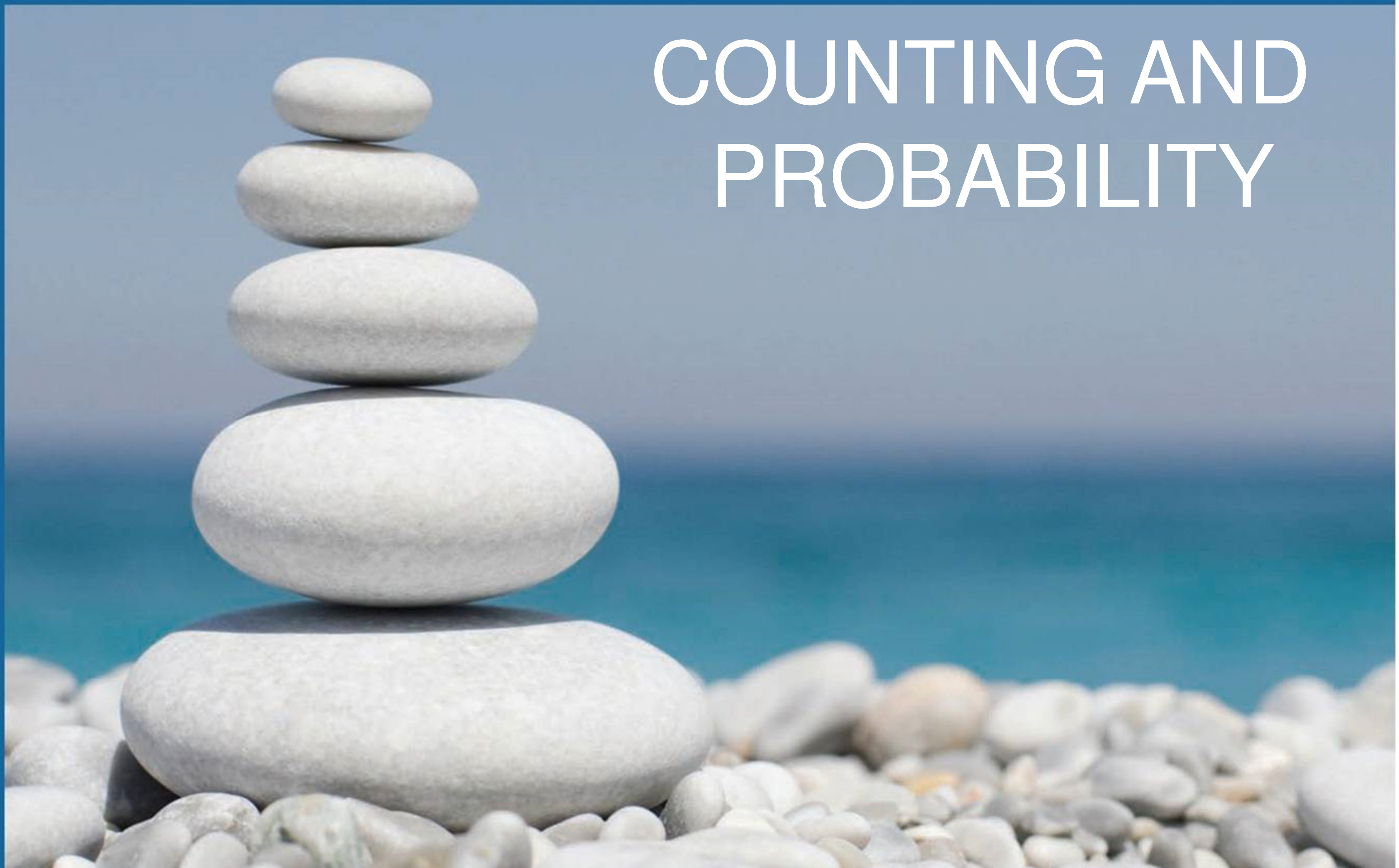
CHAPTER 9

COUNTING AND PROBABILITY



SECTION 9.1

Introduction

Introduction

Imagine tossing two coins and observing whether 0, 1, or 2 heads are obtained. It would be natural to guess that each of these events occurs about one-third of the time, but in fact this is not the case.

Table 9.1.1 below shows actual data obtained from tossing two quarters 50 times.

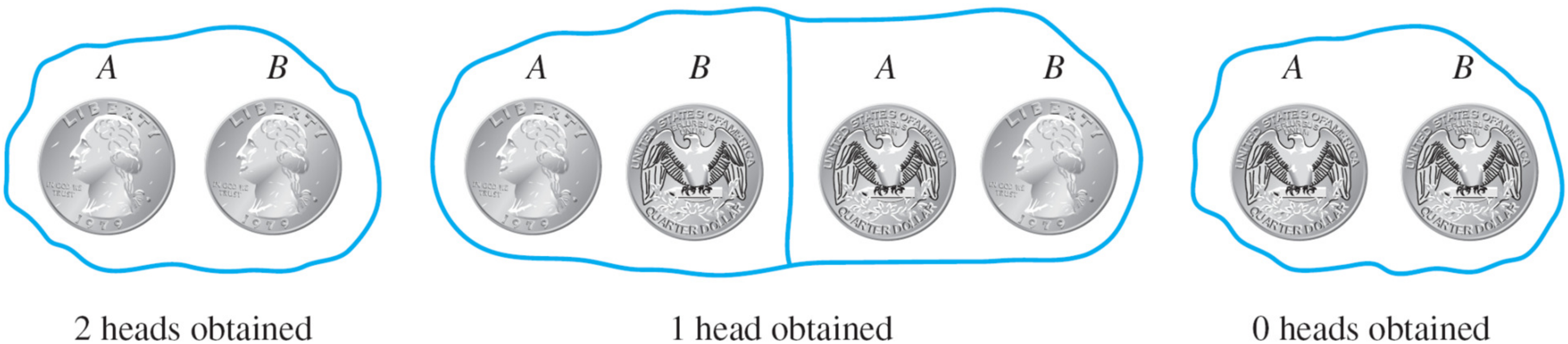
Event	Tally	Frequency (Number of times the event occurred)	Relative Frequency (Fraction of times the event occurred)
2 heads obtained	𝍸 𝍸 |	11	22%
1 head obtained	𝍸 𝍸 𝍸 𝍸 𝍸 ||	27	54%
0 heads obtained	𝍸 𝍸 ||	12	24%

Experimental Data Obtained from Tossing Two Quarters 50 Times

Table 9.1.1

Introduction

Figure 9.1.2 shows that there is a 1 in 4 chance of obtaining two heads and a 1 in 4 chance of obtaining no heads.



Equally Likely Outcomes from Tossing Two Balanced Coins

Figure 9.1.2

The chance of obtaining one head, however, is 2 in 4 because either *A* could come up heads and *B* tails or *B* could come up heads and *A* tails.

Introduction

So if you repeatedly toss two balanced coins and record the number of heads, you should expect relative frequencies similar to those shown in Table 9.1.1.

Event	Tally	Frequency (Number of times the event occurred)	Relative Frequency (Fraction of times the event occurred)
2 heads obtained	~~IIII~~ ~~IIII~~ I	11	22%
1 head obtained	~~IIII~~ ~~IIII~~ ~~IIII~~ ~~IIII~~ ~~IIII~~ II	27	54%
0 heads obtained	~~IIII~~ ~~IIII~~ II	12	24%

Experimental Data Obtained from Tossing Two Quarters 50 Times

Table 9.1.1

To formalize this analysis and extend it to more complex situations, we introduce the notions of random process, sample space, event and probability.

Introduction

To say that a process is **random** means that when it takes place, one outcome from some set of outcomes is sure to occur, but it is impossible to predict with certainty which outcome that will be.

• Definition

A **sample space** is the set of all possible outcomes of a random process or experiment. An **event** is a subset of a sample space.

In case an experiment has finitely many outcomes and all outcomes are equally likely to occur, the *probability* of an event (set of outcomes) is just the ratio of the number of outcomes in the event to the total number of outcomes.

Introduction

Equally Likely Probability Formula

If S is a finite sample space in which all outcomes are equally likely and E is an event in S, then the **probability of E**, denoted $P(E)$, is

$$P(E) = \frac{\text{the number of outcomes in } E}{\text{the total number of outcomes in } S}.$$

- **Notation**

For any finite set A, $N(A)$ denotes the number of elements in A.

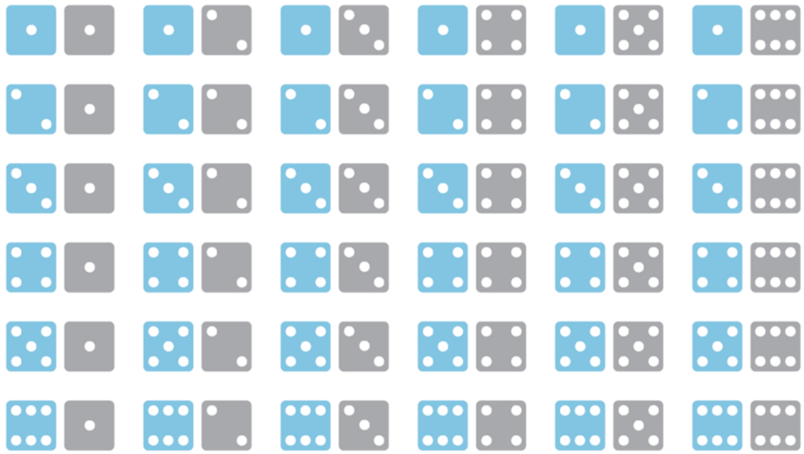
With this notation, the equally likely probability formula becomes

$$P(E) = \frac{N(E)}{N(S)}.$$

Example 2 – *Rolling a Pair of Dice*

A die is one of a pair of dice. It is a cube with six sides, each containing from one to six dots, called *pips*. Suppose a blue die and a gray die are rolled together, and the numbers of dots that occur face up on each are recorded.

The possible outcomes can be listed as follows, where in each case the die on the left is blue and the one on the right is gray.



Example 2 – *Rolling a Pair of Dice* cont'd

A more compact notation identifies, say,  with the notation 24,  with 53, and so forth.

a. Use the compact notation to write the sample space *S* of possible outcomes.

b. Use set notation to write the event *E* that the numbers showing face up have a sum of 6 and find the probability of this event.

Example 2 – *Solution*

a. S = {11, 12, 13, 14, 15, 16, 21, 22, 23, 24, 25, 26, 31, 32, 33, 34, 35, 36, 41, 42, 43, 44, 45, 46, 51, 52, 53, 54, 55, 56, 61, 62, 63, 64, 65, 66}.

b. E = {15, 24, 33, 42, 51}.

Let $P(E)$ be the probability of the event of the numbers whose sum is 6.

$$P(E) = \frac{N(E)}{N(S)} = \frac{5}{36}.$$

Counting the Elements of a List

Counting the Elements of a List

Some counting problems are as simple as counting the elements of a list.

For instance, how many integers are there from 5 through 12? To answer this question, imagine going along the list of integers from 5 to 12, counting each in turn.

list:	5	6	7	8	9	10	11	12
	↕	↕	↕	↕	↕	↕	↕	↕
count:	1	2	3	4	5	6	7	8

So the answer is 8.

Counting the Elements of a List

More generally, if m and n are integers and $m \leq n$, how many integers are there from m through n? To answer this question, note that $n = m + (n - m)$, where $n - m \geq 0$ [*since* $n \geq m$].

Theorem 9.1.1 The Number of Elements in a List

If m and n are integers and $m \leq n$, then there are $n - m + 1$ integers from m to n inclusive.

Example 4 – *Counting the Elements of a Sublist*

a. How many three-digit integers (integers from 100 to 999 inclusive) are divisible by 5?

b. What is the probability that a randomly chosen three-digit integer is divisible by 5?

Solution:

a. Imagine writing the three-digit integers in a row, noting those that are multiples of 5 and drawing arrows between each such integer and its corresponding multiple of 5.

Example 4 – *Solution*

cont'd

100	101	102	103	104	105	106	107	108	109	110	...	994	995	996	997	998	999
↕					↕					↕			↕				
$5 \cdot 20$					$5 \cdot 21$					$5 \cdot 22$			$5 \cdot 199$				

From the sketch it is clear that there are as many three-digit integers that are multiples of 5 as there are integers from 20 to 199 inclusive.

By Theorem 9.1.1, there are 199 – 20 + 1, or 180, such integers.

Hence there are 180 three-digit integers that are divisible by 5.

Example 4 – *Solution*

cont'd

b. By Theorem 9.1.1 the total number of integers from 100 through 999 is 999 – 100 + 1 = 900. By part (a), 180 of these are divisible by 5.

Hence the probability that a randomly chosen three-digit integer is divisible by 5 is 180/900 = 1/5.