CHAPTER 8

RELATIONS

SECTION 8.5

Partial Order Relations

Partial Order Relations

In order to obtain a degree in computer science at a certain university, a student must take a specified set of required courses, some of which must be completed before others can be started.

Given the prerequisite structure of the program, one might ask what is the least number of school terms needed to fulfill the degree requirements, or what is the maximum number of courses that can be taken in the same term, or whether there is a sequence in which a part-time student can take the courses one per term.

Antisymmetry

Antisymmetry

We have defined three properties of relations: reflexivity, symmetry, and transitivity. A fourth property of relations is called *antisymmetry*.

In terms of the arrow diagram of a relation, saying that a relation is antisymmetric is the same as saying that whenever there is an arrow going from one element to another *distinct* element, there is *not* an arrow going back from the second to the first.

Antisymmetry

- **Definition**

Let R be a relation on a set A. R is **antisymmetric** if, and only if,

$$\text{for all } a \text{ and } b \text{ in } A, \quad \text{if } a\ R\ b \text{ and } b\ R\ a \text{ then } a = b.$$

By taking the negation of the definition, you can see that a relation R is ***not*** **antisymmetric** if, and only if,

there are elements a and b in A such that $a\ R\ b$ and $b\ R\ a$ but $a \neq b$.

Example 2 – *Testing for Antisymmetry of "Divides" Relations*

Let R_1 be the "divides" relation on the set of all positive integers, and let R_2 be the "divides" relation on the set of all integers.

$$\text{For all } a, b \in Z^+, \qquad a \; R_1 \; b \quad \Leftrightarrow \quad a \mid b.$$

$$\text{For all } a, b \in Z, \qquad a \; R_2 \; b \quad \Leftrightarrow \quad a \mid b.$$

a. Is R_1 antisymmetric? Prove or give a counterexample.

b. Is R_2 antisymmetric? Prove or give a counterexample.

Example 2 – *Solution*

a. R_1 is antisymmetric.

Proof:
Suppose a and b are positive integers such that $a\ R_1\ b$ and $b\ R_1\ a$. *[We must show that $a = b$.]* By definition of R_1, $a \mid b$ and $b \mid a$.

Thus, by definition of divides, there are integers k_1 and k_2 with $b = k_1 a$ and $a = k_2 b$. It follows that

$$b = k_1 a = k_1(k_2 b) = (k_1 k_2) b.$$

Dividing both sides by b gives

$$k_1 k_2 = 1.$$

Example 2 – *Solution*

cont'd

Now since a and b are both integers k_1 and k_2 are both positive integers also.

But the only product of two positive integers that equals 1 is 1 • 1.

Thus

$$k_1 = k_2 = 1$$

and so

$$a = k_2 b = 1 \cdot b = b.$$

[This is what was to be shown.]

Example 2 – *Solution*

cont'd

b. R_2 is not antisymmetric.

Counterexample:

Let $a = 2$ and $b = –2$. Then $a \mid b$ *[since* $–2 = (–1) \cdot 2$*]* and $b \mid a$ *[*since $2 = (–1)(–2)$*]*.

Hence $a\ R_2\ b$ and $b\ R_2\ a$ but $a \neq b$.

Partial Order Relations

Partial Order Relations

A relation that is reflexive, antisymmetric, and transitive is called a *partial order*.

• Definition

Let R be a relation defined on a set A. R is a **partial order relation** if, and only if, R is reflexive, antisymmetric, and transitive.

Two fundamental partial order relations are the "less than or equal to" relation on a set of real numbers and the "subset" relation on a set of sets.

These can be thought of as models, or paradigms, for general partial order relations.

Example 4 – *A "Divides" Relation on a Set of Positive Integers*

Let | be the "divides" relation on a set *A* of positive integers. That is, for all $a, b \in A$,

$$a \mid b \quad \Leftrightarrow \quad b = ka \text{ for some integer } k.$$

Prove that | is a partial order relation on *A*.

Solution:

| is reflexive: *[We must show that for all $a \in A$, $a \mid a$.]*

Suppose $a \in A$. Then $a = 1 \cdot a$, so $a \mid a$ by definition of divisibility.

Example 4 – *Solution*

cont'd

| is antisymmetric: *[We must show that for all a, b* $\in$ *A, if a | b and b | a then a = b.]* The proof of this is virtually identical to that of Example 2(a).

/ is transitive: To show transitivity means to show that for all *a*, *b*, *c* $\in$ *A*, if *a* | *b* and *b* | *c* then *a* | *c*. But this was proved as Theorem 4.3.3.

Theorem 4.3.3 Transitivity of Divisibility

For all integers a, b, and c, if a divides b and b divides c, then a divides c.

Since | is reflexive, antisymmetric, and transitive, | is a partial order relation on *A*.

Partial Order Relations

• Notation

Because of the special paradigmatic role played by the $\leq$ relation in the study of partial order relations, the symbol $\preceq$ is often used to refer to a general partial order relation, and the notation $x \preceq y$ is read "x is less than or equal to y" or "y is greater than or equal to x."

Lexicographic Order

Lexicographic Order

To figure out which of two words comes first in an English dictionary, you compare their letters one by one from left to right. If all letters have been the same to a certain point and one word runs out of letters, that word comes first in the dictionary.

For example, *play* comes before *playhouse*. If all letters up to a certain point are the same and the next letters differ, then the word whose next letter is located earlier in the alphabet comes first in the dictionary. For instance, *playhouse* comes *before playmate*.

Lexicographic Order

More generally, if *A* is any set with a partial order relation, then a *dictionary* or *lexicographic* order can be defined on a set of strings over *A* as indicated in the following theorem.

Theorem 8.5.1

Let A be a set with a partial order relation R, and let S be a set of strings over A. Define a relation $\preceq$ on S as follows:

For any two strings in S, $a_1 a_2 \cdots a_m$ and $b_1 b_2 \cdots b_n$, where m and n are positive integers,

1. If $m \le n$ and $a_i = b_i$ for all $i = 1, 2, \ldots, m$, then

$$a_1 a_2 \cdots a_m \preceq b_1 b_2 \cdots b_n.$$

2. If for some integer k with $k \le m$, $k \le n$, and $k \ge 1$, $a_i = b_i$ for all $i = 1, 2, \ldots, k-1$, and $a_k \neq b_k$, but $a_k \, R \, b_k$ then

$$a_1 a_2 \cdots a_m \preceq b_1 b_2 \cdots b_n.$$

3. If ε is the null string and s is any string in S, then $\epsilon \preceq s$.

If no strings are related other than by these three conditions, then $\preceq$ is a partial order relation.

Lexicographic Order

• Definition

The partial order relation of Theorem 8.5.1 is called the **lexicographic order for** S that corresponds to the partial order R on A.

Example 6 – *A Lexicographic Order*

Let $A = \{x, y\}$ and let R be the following partial order relation on A:

$$R = \{(x, x), (x, y), (y, y)\}.$$

Let S be the set of all strings over A, and denote by $\preceq$ the lexicographic order for S that corresponds to R.

a. Is $x \preceq xx$? $x \preceq xy$? $xx \preceq xxx$? $yxy \preceq yxyxxx$?

b. Is $x \preceq y$? $xx \preceq xyx$? $xxxy \preceq xy$? $yxyxxyy \preceq yxyxy$?

c. Is $\epsilon \preceq x$? $\epsilon \preceq xy$? $\epsilon \preceq yyxy$?

Example 6 – *Solution*

a. Yes in all cases, by relation (1) in theorem 8.5.1.

b. Yes in all cases, by relation (2) in theorem 8.5.1.

c. Yes in all cases, by relation (3) in theorem 8.5.1.

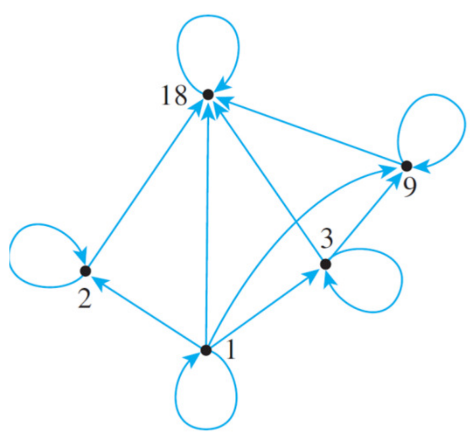
Hasse Diagrams

Hasse Diagrams

Let $A = \{1, 2, 3, 9, 18\}$ and consider the "divides" relation on A: For all $a, b \in A$,

$$a \mid b \quad \Leftrightarrow \quad b = ka \text{ for some integer } k.$$

The directed graph of this relation has the following appearance:



Hasse Diagrams

Note that there is a loop at every vertex, all other arrows point in the same direction (upward), and any time there is an arrow from one point to a second and from the second point to a third, there is an arrow from the first point to the third.

Given any partial order relation defined on a finite set, it is possible to draw the directed graph in such a way that all of these properties are satisfied.

Hasse Diagrams

This makes it possible to associate a somewhat simpler graph, called a **Hasse diagram** (after Helmut Hasse, a twentieth-century German number theorist), with a partial order relation defined on a finite set.

To obtain a Hasse diagram, proceed as follows:

Start with a directed graph of the relation, placing vertices on the page so that all arrows point upward. Then eliminate

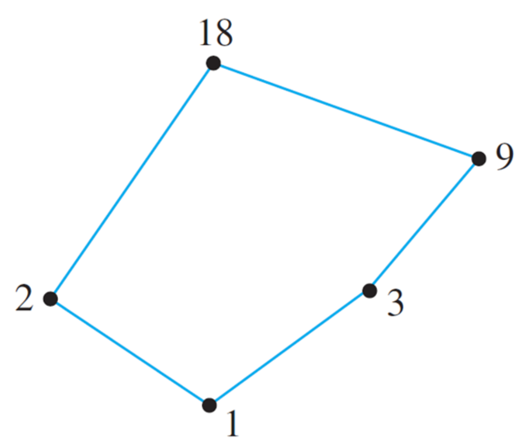
1. the loops at all the vertices,

Hasse Diagrams

2. all arrows whose existence is implied by the transitive property,

3. the direction indicators on the arrows.

For the relation given previously, the Hasse diagram is as follows:



Example 7 – *Constructing a Hasse Diagram*

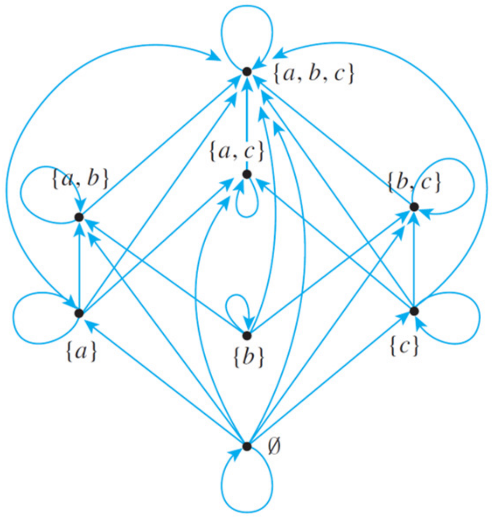
Consider the "subset" relation, $\subseteq$, on the set $\mathscr{P}(\{a, b, c\})$. That is, for all sets U and V in $\mathscr{P}(\{a, b, c\})$,

$$U \subseteq V \quad \Leftrightarrow \quad \forall x, \text{ if } x \in U \text{ then } x \in V.$$

Construct the Hasse diagram for this relation.

Solution:

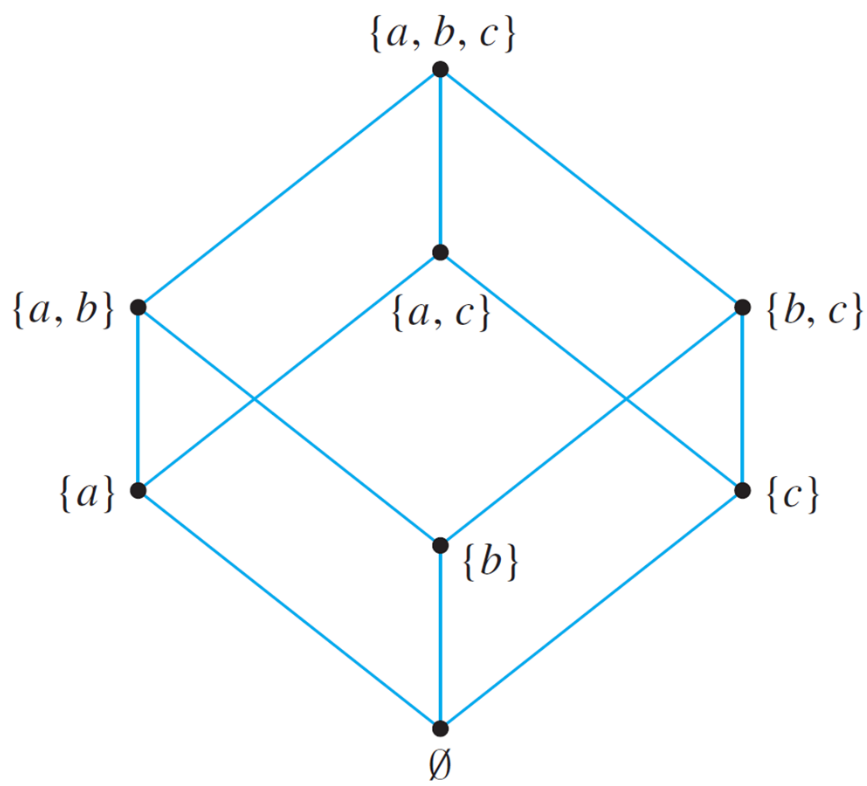
Draw the directed graph of the relation in such a way that all arrows except loops point upward.



Example 7 – *Solution*

cont'd

Then strip away all loops, unnecessary arrows, and direction indicators to obtain the Hasse diagram.



Hasse Diagrams

To recover the directed graph of a relation from the Hasse diagram, just reverse the instructions given previously, using the knowledge that the original directed graph was sketched so that all arrows pointed upward:

1. Reinsert the direction markers on the arrows making all arrows point upward.

2. Add loops at each vertex.

3. For each sequence of arrows from one point to a second and from that second point to a third, add an arrow from the first point to the third.

Partially and Totally Ordered Sets

Partially and Totally Ordered Sets

Given any two real numbers x and y, either $x \leq y$ or $y \leq x$. In a situation like this, the elements x and y are said to be *comparable*.

On the other hand, given two subsets A and B of $\{a, b, c\}$, it may be the case that neither $A \subseteq B$ nor $B \subseteq A$.

For instance, let $A = \{a, b\}$ and $B = \{b, c\}$. Then $A \nsubseteq B$ and $B \nsubseteq A$.

In such a case, A and B are said to be *noncomparable*.

Partially and Totally Ordered Sets

- **Definition**

Suppose $\preceq$ is a partial order relation on a set A. Elements a and b of A are said to be **comparable** if, and only if, either $a \preceq b$ or $b \preceq a$. Otherwise, a and b are called **noncomparable.**

When all the elements of a partial order relation are comparable, the relation is called a *total order*.

- **Definition**

If R is a partial order relation on a set A, and for any two elements a and b in A either $a \; R \; b$ or $b \; R \; a$, then R is a **total order relation** on A.

Partially and Totally Ordered Sets

Both the “less than or equal to” relation on sets of real numbers and the lexicographic order of the set of words in a dictionary are total order relations.

Note that the Hasse diagram for a total order relation can be drawn as a single vertical “chain.”

Many important partial order relations have elements that are not comparable and are, therefore, not total order relations.

Partially and Totally Ordered Sets

For instance, the subset relation on $\mathscr{P}(\{a, b, c\})$ is not a total order relation because, as shown previously, the subsets $\{a, b\}$ and $\{a, c\}$ of $\{a, b, c\}$ are not comparable.

In addition, a "divides" relation is not a total order relation unless the elements are all powers of a single integer.

A set A is called a **partially ordered set** (or **poset**) with respect to a relation $\preceq$ if, and only if, $\preceq$ is a partial order relation on A.

Partially and Totally Ordered Sets

For instance, the set of real numbers is a partially ordered set with respect to the "less than or equal to" relation $\leq$, and a set of sets is partially ordered with respect to the "subset" relation $\subseteq$.

It is entirely straightforward to show that *any subset of a partially ordered set is partially ordered*.

This, of course, assumes the "same definition" for the relation on the subset as for the set as a whole. A set *A* is called a **totally ordered set** with respect to a relation $\preceq$ if, and only if, *A* is partially ordered with respect to $\preceq$ and $\preceq$ is a total order.

Partially and Totally Ordered Sets

A set that is partially ordered but not totally ordered may have totally ordered subsets. Such subsets are called *chains*.

• Definition

Let A be a set that is partially ordered with respect to a relation $\preceq$. A subset B of A is called a **chain** if, and only if, the elements in each pair of elements in B is comparable. In other words, $a \preceq b$ or $b \preceq a$ for all a and b in A. The **length of a chain** is one less than the number of elements in the chain.

Observe that if *B* is a chain in *A*, then *B* is a totally ordered set with respect to the "restriction" of $\preceq$ to *B*.

Example 9 – *A Chain of Subsets*

The set $\mathscr{P}(\{a, b, c\})$ is partially ordered with respect to the subset relation. Find a chain of length 3 in $\mathscr{P}(\{a, b, c\})$.

Solution:

Since $\emptyset \subseteq \{a\} \subseteq \{a, b, \} \subseteq \{a, b, c\}$, the set

$$S = \{\emptyset, \{a\}, \{a, b\}, \{a, b, c\}\}$$

is a chain of length 3 in $\mathscr{P}(\{a, b, c\})$.

Partially and Totally Ordered Sets

A *maximal element* in a partially ordered set is an element that is greater than or equal to every element to *which it is comparable*. (There may be many elements to which it is *not* comparable.)

A *greatest element* in a partially ordered set is an element that is greater than or equal to *every* element in the set (so it is comparable to every element in the set). Minimal and least elements are defined similarly.

Partially and Totally Ordered Sets

• Definition

Let a set A be partially ordered with respect to a relation $\preceq$.

1. An element a in A is called a **maximal element of A** if, and only if, for all b in A, either $b \preceq a$ or b and a are not comparable.
2. An element a in A is called a **greatest element of A** if, and only if, for all b in A, $b \preceq a$.
3. An element a in A is called a **minimal element of A** if, and only if, for all b in A, either $a \preceq b$ or b and a are not comparable.
4. An element a in A is called a **least element of A** if, and only if, for all b in A, $a \preceq b$.

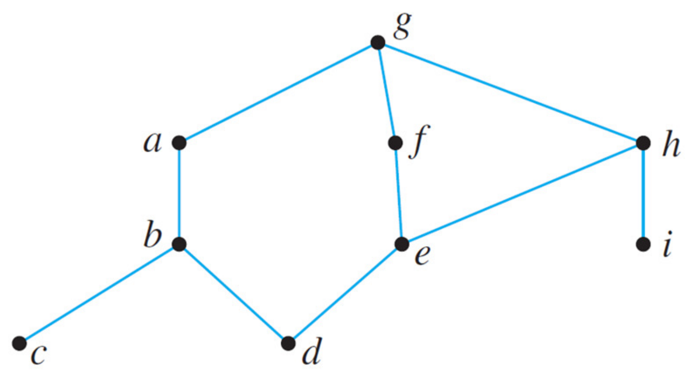
Partially and Totally Ordered Sets

A greatest element is maximal, but a maximal element need not be a greatest element. However, every finite subset of a totally ordered set has both a least element and a greatest element.

Similarly, a least element is minimal, but a minimal element need not be a least element. Furthermore, a set that is partially ordered with respect to a relation can have at most one greatest element and one least element, but it may have more than one maximal or minimal element.

Example 10 – *Maximal, Minimal, Greatest, and Least Elements*

Let $A = \{a, b, c, d, e, f, g, h, i\}$ have the partial ordering $\preceq$ defined by the following Hasse diagram. Find all maximal, minimal, greatest, and least elements of *A*.



Solution:

There is just one maximal element, *g*, which is also the greatest element. The minimal elements are *c*, *d*, and *i*, and there is no least element.

Topological Sorting

Topological Sorting

Is it possible to input the sets of $\mathscr{P}(\{a, b, c\})$ into a computer in a way that is *compatible* with the subset relation $\subseteq$ in the sense that if set *U* is a subset of set *V*, then *U* is input before *V*?

The answer, as it turns out, is yes. For instance, the following input order satisfies the given condition:

$$\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}.$$

Another input order that satisfies the condition is

$$\emptyset, \{a\}, \{b\}, \{a, b\}, \{c\}, \{b, c\}, \{a, c\}, \{a, b, c\}.$$

Topological Sorting

- **Definition**

Given partial order relations $\preceq$ and $\preceq'$ on a set A, $\preceq'$ is **compatible** with $\preceq$ if, and only if, for all a and b in A, if $a \preceq b$ then $a \preceq' b$.

Given an arbitrary partial order relation $\preceq$ on a set *A*, is there a total order $\preceq'$ on *A* that is compatible with $\preceq$? If the set on which the partial order is defined is finite, then the answer is yes. A total order that is compatible with a given order is called a *topological sorting*.

- **Definition**

Given partial order relations $\preceq$ and $\preceq'$ on a set A, $\preceq'$ is a **topological sorting** for $\preceq$ if, and only if, $\preceq'$ is a total order that is compatible with $\preceq$.

Topological Sorting

Constructing a Topological Sorting

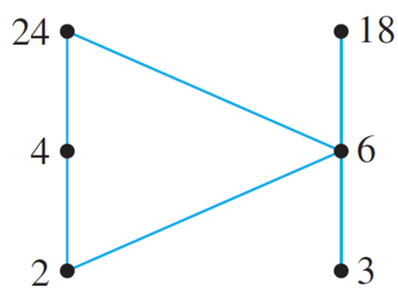
Let $\preceq$ be a partial order relation on a nonempty finite set A. To construct a topological sorting,

1. Pick any minimal element x in A. *[Such an element exists since A is nonempty.]*
2. Set $A' := A - \{x\}$.
3. Repeat steps *a–c* while $A' \neq \emptyset$.
   a. Pick any minimal element y in A'.
   b. Define $x \preceq' y$.
   c. Set $A' := A' - \{y\}$ and $x := y$.
   [Completion of steps 1–3 of this algorithm gives enough information to construct the Hasse diagram for the total ordering $\preceq'$. We have already shown how to use the Hasse diagram to obtain a complete directed graph for a relation.]

Example 11 – *A Topological Sorting*

Consider the set $A = \{2, 3, 4, 6, 18, 24\}$ ordered by the "divides" relation $|$.

The Hasse diagram of this relation is the following:



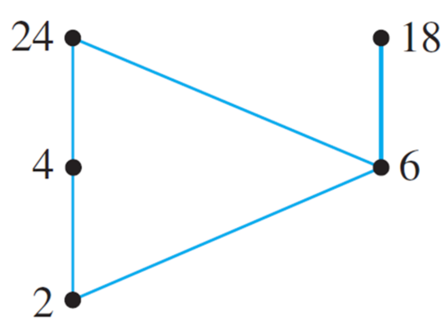
The ordinary "less than or equal to" relation $\leq$ on this set is a topological sorting for it since for positive integers a and b, if $a \mid b$ then $a \leq b$. Find another topological sorting for this set.

Example 11 – *Solution*

The set has two minimal elements: 2 and 3. Either one may be chosen; say you pick 3. The beginning of the total order is

total order: 3.

Set $A' = A - \{3\}$. You can indicate this by removing 3 from the Hasse diagram as shown below.



Example 11 – *Solution*

cont'd

Next choose minimal element from $A' - \{3\}$. Only 2 is minimal, so you must pick it. The total order thus far is

total order: $3 \preceq 2$.

Set $A' = (A - \{3\}) - \{2\} = A - \{3, 2\}$.

You can indicate this by removing 2 from the Hasse diagram, as is shown below.

Choose a minimal element from $A' - \{3, 2\}$.

Example 11 – *Solution*

cont'd

Again you have two choices: 4 and 6. Say you pick 6. The total order for the elements chosen thus far is

$$\text{total order: } 3 \preceq 2 \preceq 6.$$

You continue in this way until every element of *A* has been picked. One possible sequence of choices gives

$$\text{total order: } 3 \preceq 2 \preceq 6 \preceq 18 \preceq 4 \preceq 24.$$

Example 11 – *Solution*

cont'd

You can verify that this order is compatible with the "divides" partial order by checking that for each pair of elements a and b in A such that $a \mid b$, then $a \preceq b$.

Note that it is *not* the case that if $a \preceq b$ then $a \mid b$.

An Application

An Application

To return to the example that introduced this section, note that the following defines a partial order relation on the set of courses required for a university degree: For all required courses *x* and *y*,

$$x \preceq y \quad \Leftrightarrow \quad x = y \quad \text{or} \quad x \text{ is a prerequisite for } y$$

If the Hasse diagram for the relation is drawn, then the questions raised at the beginning of this section can be answered easily.

An Application

For instance, consider the Hasse diagram for the requirements at a particular university, which is shown in Figure 8.5.1.

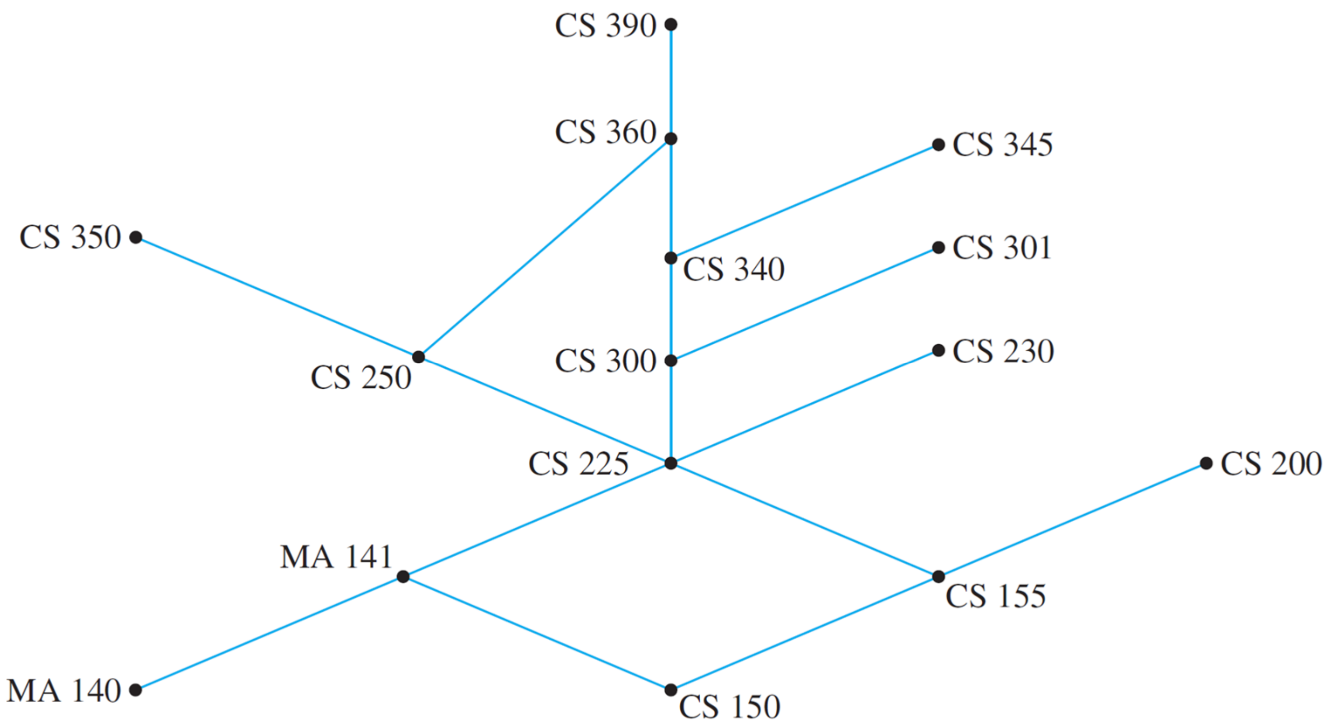


Figure 8.5.1

An Application

The minimum number of school terms needed to complete the requirements is the size of a longest chain, which is 7 (150, 155, 225, 300, 340, 360, 390, for example).

The maximum number of courses that could be taken in the same term (assuming the university allows it) is the maximum number of noncomparable courses, which is 6 (350, 360, 345, 301, 230, 200, for example).

An Application

A part-time student could take the courses in a sequence determined by constructing a topological sorting for the set.

(One such sorting is 140, 150, 141, 155, 200, 225, 230, 300, 250, 301, 340, 345, 350, 360, 390. There are many others.)

PERT and CPM

PERT and CPM

Two important and widely used applications of partial order relations are **PERT** (Program Evaluation and Review Technique) and **CPM** (Critical Path Method).

These techniques came into being in the 1950s as planners came to grips with the complexities of scheduling the individual activities needed to complete very large projects, and although they are very similar, their developments were independent.

PERT and CPM

PERT was developed by the U.S.

Navy to help organize the construction of the Polaris submarine, and CPM was developed by the E. I. Du Pont de Nemours company for scheduling chemical plant maintenance.

Here is a somewhat simplified example of the way the techniques work.

Example 12 – *A Job Scheduling Problem*

At an automobile assembly plant, the job of assembling an automobile can be broken down into these tasks:

1. Build frame.

2. Install engine, power train components, gas tank.

3. Install brakes, wheels, tires.

4. Install dashboard, floor, seats.

Example 12 – *A Job Scheduling Problem* cont'd

5. Install electrical lines.

6. Install gas lines.

7. Install brake lines.

8. Attach body panels to frame.

9. Paint body.

Certain of these tasks can be carried out at the same time, whereas some cannot be started until other tasks are finished.

Example 12 – *A Job Scheduling Problem* cont'd

Table 8.5.1 summarizes the order in which tasks can be performed and the time required to perform each task.

Task	Immediately Preceding Tasks	Time Needed to Perform Task
1		7 hours
2	1	6 hours
3	1	3 hours
4	2	6 hours
5	2, 3	3 hours
6	4	1 hour
7	2, 3	1 hour
8	4, 5	2 hours
9	6, 7, 8	5 hours

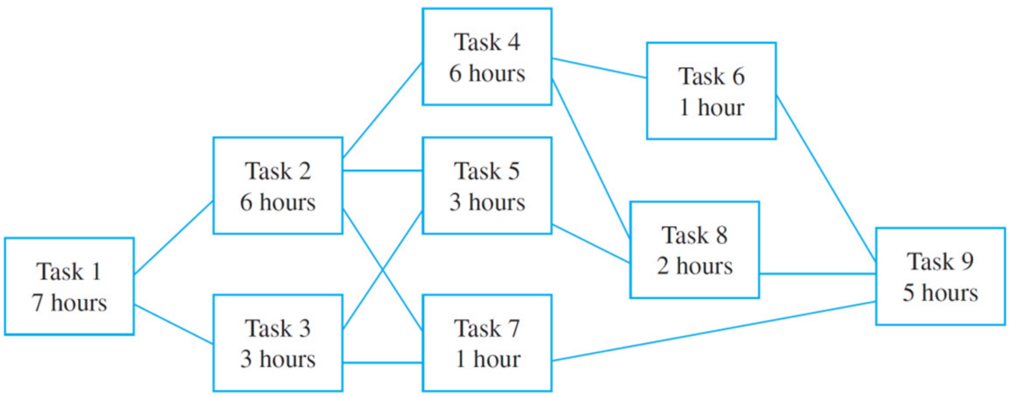
Table 8.5.1

Example 12 – *A Job Scheduling Problem* cont'd

Let *T* be the set of all tasks, and consider the partial order relation $\preceq$ defined on *T* as follows: For all tasks *x* and *y* in *T*,

$$x \preceq y \quad \Leftrightarrow \quad x = y \text{ or } x \text{ precedes } y.$$

If the Hasse diagram of this relation is turned sideways (as is customary in PERT and CPM analysis), it has the appearance shown below.



Example 12 – *A Job Scheduling Problem* cont'd

What is the minimum time required to assemble a car? You can determine this by working from left to right across the diagram, noting for each task (say, just above the box representing that task) the minimum time needed to complete that task starting from the beginning of the assembly process.

For instance, you can put a 7 above the box for task 1 because task 1 requires 7 hours.

Task 2 requires completion of task 1 (7 hours) plus 6 hours for itself, so the minimum time required to complete task 2, starting at the beginning of the assembly process, is
$7 + 6 = 13$ hours.

Example 12 – *A Job Scheduling Problem* cont'd

You can put a 13 above the box for task 2.

Similarly, you can put a 10 above the box for task 3 because 7 + 3 = 10.

Now consider what number you should write above the box for task 5.

The minimum times to complete tasks 2 and 3, starting from the beginning of the assembly process, are 13 and 10 hours respectively.

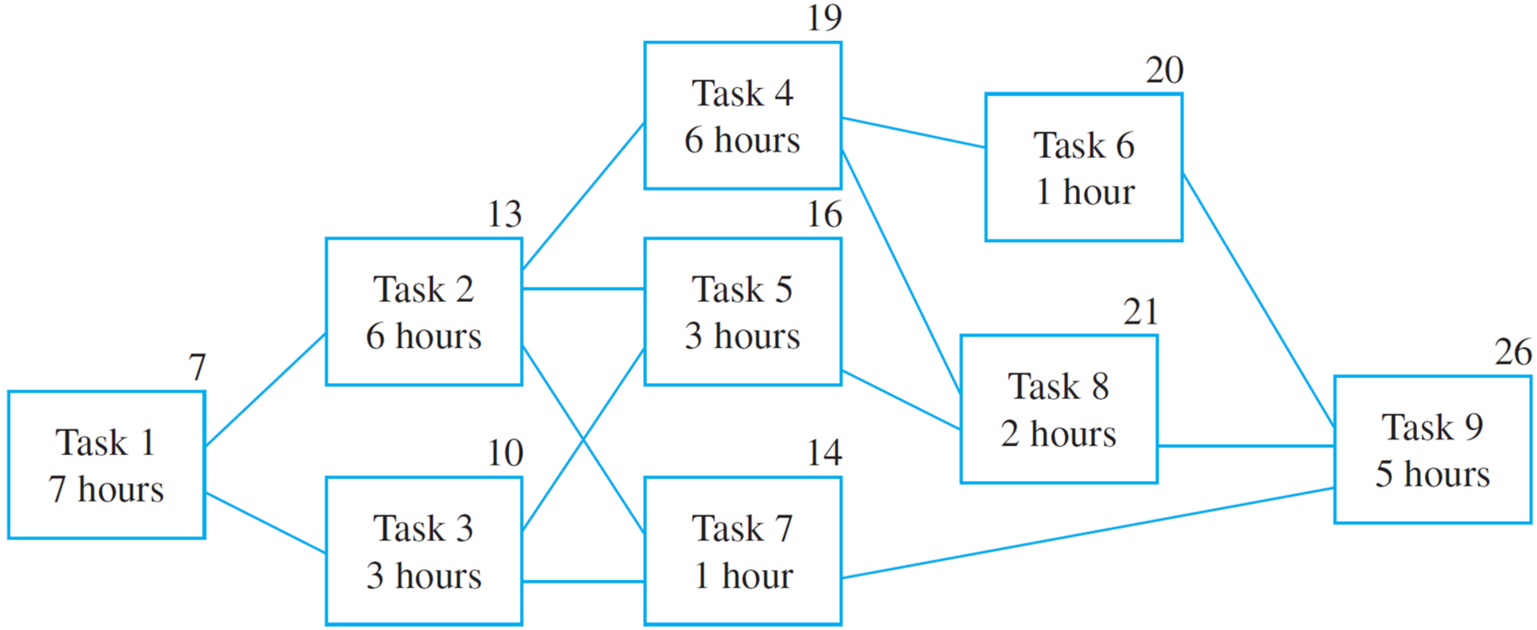
Example 12 – *A Job Scheduling Problem* cont'd

Since *both* tasks must be completed before task 5 can be started, the minimum time to complete task 5, starting from the beginning, is the time needed for task 5 itself (3 hours) plus the *maximum* of the times to complete tasks 2 and 3 (13 hours), and this equals 3 + 13 = 16 hours.

Thus you should place the number 16 above the box for task 5. The same reasoning leads you to place a 14 above the box for task 7.

Example 12 – *A Job Scheduling Problem* cont'd

Similarly, you can place a 19 above the box for task 4, a 20 above the box for task 6, a 21 above the box for task 8, and a 26 above the box for task 9, as shown below.



Example 12 – *A Job Scheduling Problem* cont'd

This analysis shows that at least 26 hours are required to complete task 9 starting from the beginning of the assembly process. When task 9 is finished, the assembly is complete, so 26 hours is the minimum time needed to accomplish the whole process.

Note that the minimum time required to complete tasks 1, 2, 4, 8, and 9 in sequence is exactly 26 hours.

This means that a delay in performing any one of these tasks causes a delay in the total time required for assembly of the car.

For this reason, the path through tasks 1, 2, 4, 8, and 9 is called a **critical path.**