**CHAPTER 8**

# RELATIONS

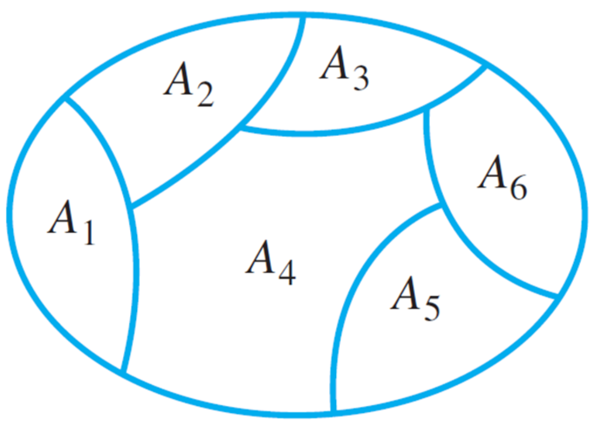
**SECTION 8.3**

# Equivalence Relations

# The Relation Induced by a Partition

# The Relation Induced by a Partition

A **partition** of a set *A* is a finite or infinite collection of nonempty, mutually disjoint subsets whose union is *A*. The diagram of Figure 8.3.1 illustrates a partition of a set *A* by subsets $A_1, A_2, \ldots, A_6$.




$$A_i \cap A_j = \emptyset, \text{ whenever } i \neq j$$
$$A_i \cup A_2 \cup \cdots \cup A_6 = A$$

A Partition of a Set

**Figure 8.3.1**

# The Relation Induced by a Partition

**• Definition**

Given a partition of a set $A$, the **relation induced by the partition,** $R$, is defined on $A$ as follows: For all $x, y \in A$,

$$x \; R \; y \quad \Leftrightarrow \quad \text{there is a subset } A_i \text{ of the partition such that both } x \text{ and } y \text{ are in } A_i.$$

# Example 1 – *Relation Induced by a Partition*

Let $A = \{0, 1, 2, 3, 4\}$ and consider the following partition of $A$:

$$\{0, 3, 4\}, \{1\}, \{2\}.$$

Find the relation $R$ induced by this partition.

Solution:

Since $\{0, 3, 4\}$ is a subset of the partition,

$0 \; R \; 3$ because both 0 and 3 are in $\{0, 3, 4\}$,

$3 \; R \; 0$ because both 3 and 0 are in $\{0, 3, 4\}$,

# Example 1 – *Solution*

cont'd

$0 \; R \; 4$ because both 0 and 4 are in {0, 3, 4},

$4 \; R \; 0$ because both 4 and 0 are in {0, 3, 4},

$3 \; R \; 4$ because both 3 and 4 are in {0, 3, 4}, and

$4 \; R \; 3$ because both 4 and 3 are in {0, 3, 4}.

Also, $0 \; R \; 0$ because both 0 and 0 are in {0, 3, 4}

$3 \; R \; 3$ because both 3 and 3 are in {0, 3, 4}, and

$4 \; R \; 4$ because both 4 and 4 are in {0, 3, 4}.

# Example 1 – *Solution*

cont'd

Since {1} is a subset of the partition,

$1 \ R \ 1$ because both 1 and 1 are in {1},

and since {2} is a subset of the partition,

$2 \ R \ 2$ because both 2 and 2 are in {2}.

Hence

$R = \{(0, 0), (0, 3), (0, 4), (1, 1), (2, 2), (3, 0), (3, 3), (3, 4), (4, 0), (4, 3), (4, 4)\}$.

# The Relation Induced by a Partition

The fact is that a relation induced by a partition of a set satisfies all three properties: reflexivity, symmetry, and transitivity.

**Theorem 8.3.1**

Let $A$ be a set with a partition and let $R$ be the relation induced by the partition. Then $R$ is reflexive, symmetric, and transitive.

# Definition of an Equivalence Relation

# Definition of an Equivalence Relation

A relation on a set that satisfies the three properties of reflexivity, symmetry, and transitivity is called an *equivalence relation*.

- **Definition**

Let $A$ be a set and $R$ a relation on $A$. $R$ is an **equivalence relation** if, and only if, $R$ is reflexive, symmetric, and transitive.

Thus, according to Theorem 8.3.1, the relation induced by a partition is an equivalence relation.

## Example 2 – *An Equivalence Relation on a Set of Subsets*

Let $X$ be the set of all nonempty subsets of {1, 2, 3}. Then

$$X = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$$

Define a relation **R** on $X$ as follows: For all $A$ and $B$ in $X$,

$A$ **R** $B \Leftrightarrow$ the least element of $A$ equals the least element of $B$.

Prove that **R** is an equivalence relation on $X$.

# Example 2 – *Solution*

**R is reflexive:** Suppose *A* is a nonempty subset of {1, 2, 3}. *[We must show that A* **R** *A.]*

It is true to say that the least element of *A* equals the least element of *A*. Thus, by definition of *R*, *A* **R** *A*.

**R is symmetric:** Suppose *A* and *B* are nonempty subsets of {1, 2, 3} and *A* **R** *B*. *[We must show that B* **R** *A.]*

Since *A* **R** *B*, the least element of *A* equals the least element of *B*.

But this implies that the least element of *B* equals the least element of *A*, and so, by definition of **R**, *B* **R** *A*.

# Example 2 – *Solution*

cont'd

**R is transitive:** Suppose *A*, *B*, and *C* are nonempty subsets of {1, 2, 3}, *A* **R** *B*, and *B* **R** *C*. *[We must show that A* **R** *C.]*

Since *A* **R** *B*, the least element of *A* equals the least element of *B* and since *B* **R** *C*, the least element of *B* equals the least element of *C*.

Thus the least element of *A* equals the least element of *C*, and so, by definition of **R**, *A* **R** *C*.

# Equivalence Classes of an Equivalence Relation

# Equivalence Classes of an Equivalence Relation

Suppose there is an equivalence relation on a certain set. If *a* is any particular element of the set, then one can ask, "What is the subset of all elements that are related to *a*?" This subset is called the *equivalence class* of *a*.

**• Definition**

Suppose $A$ is a set and $R$ is an equivalence relation on $A$. For each element $a$ in $A$, the **equivalence class of *a*,** denoted **[*a*]** and called the **class of *a*** for short, is the set of all elements $x$ in $A$ such that $x$ is related to $a$ by $R$.

In symbols:

$$[a] = \{x \in A \mid x \; R \; a\}$$

# Equivalence Classes of an Equivalence Relation

When several equivalence relations on a set are under discussion, the notation $[a]_R$ is often used to denote the equivalence class of *a* under *R*.

The procedural version of this definition is

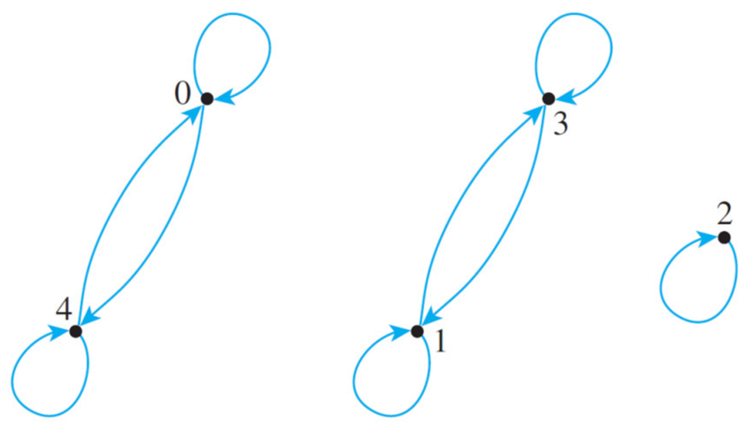
$$\text{for all } x \in A, \quad x \in [a] \quad \Leftrightarrow \quad x \; R \; a.$$

## Example 5 – *Equivalence Classes of a Relation Given as a set of Ordered Pairs*

Let $A = \{0, 1, 2, 3, 4\}$ and define a relation $R$ on $A$ as follows:

$$R = \{(0, 0), (0, 4), (1, 1), (1, 3), (2, 2), (3, 1), (3, 3), (4, 0), (4, 4)\}.$$

The directed graph for $R$ is as shown below. As can be seen by inspection, $R$ is an equivalence relation on $A$. Find the distinct equivalence classes of $R$.



# Example 5 – *Solution*

First find the equivalence class of every element of *A*.

$$[0] = \{x \in A \mid x \ R \ 0\} = \{0, 4\}$$

$$[1] = \{x \in A \mid x \ R \ 1\} = \{1, 3\}$$

$$[2] = \{x \in A \mid x \ R \ 2\} = \{2\}$$

$$[3] = \{x \in A \mid x \ R \ 3\} = \{1, 3\}$$

$$[4] = \{x \in A \mid x \ R \ 4\} = \{0, 4\}$$

Note that [0] = [4] and [1] = [3]. Thus the *distinct* equivalence classes of the relation are

{0, 4}, {1, 3}, and {2}.

# Equivalence Classes of an Equivalence Relation

The first lemma says that if two elements of $A$ are related by an equivalence relation $R$, then their equivalence classes are the same.

**Lemma 8.3.2**

Suppose $A$ is a set, $R$ is an equivalence relation on $A$, and $a$ and $b$ are elements of $A$. If $a\ R\ b$, then $[a] = [b]$.

This lemma says that if a certain condition is satisfied, then $[a] = [b]$. Now $[a]$ and $[b]$ are *sets*, and two sets are equal if, and only if, each is a subset of the other.

# Equivalence Classes of an Equivalence Relation

Hence the proof of the lemma consists of two parts: first, a proof that $[a] \subseteq [b]$ and second, a proof that $[b] \subseteq [a]$.

To show each subset relation, it is necessary to show that every element in the left-hand set is an element of the right-hand set.

The second lemma says that any two equivalence classes of an equivalence relation are either mutually disjoint or identical.

**Lemma 8.3.3**

If $A$ is a set, $R$ is an equivalence relation on $A$, and $a$ and $b$ are elements of $A$, then

$$\text{either} \quad [a] \cap [b] = \emptyset \quad \text{or} \quad [a] = [b].$$

# Equivalence Classes of an Equivalence Relation

The statement of Lemma 8.3.3 has the form

if $p$ then ($q$ or $r$),

where $p$ is the statement "$A$ is a set, $R$ is an equivalence relation on $A$, and $a$ and $b$ are elements of $A$," $q$ is the statement "$[a] \cap [b] = \emptyset$," and $r$ is the statement "$[a] = [b]$."

**Theorem 8.3.4 The Partition Induced by an Equivalence Relation**

If $A$ is a set and $R$ is an equivalence relation on $A$, then the distinct equivalence classes of $R$ form a partition of $A$; that is, the union of the equivalence classes is all of $A$, and the intersection of any two distinct classes is empty.

## Congruence Modulo *n*

## Example 10 – *Equivalence Classes of Congruence Modulo* 3

Let *R* be the relation of congruence modulo 3 on the set **Z** of all integers. That is, for all integers *m* and *n*,

$$m \; R \; n \quad \Leftrightarrow \quad 3 \mid (m - n) \quad \Leftrightarrow \quad m \equiv n \pmod 3.$$

Describe the distinct equivalence classes of *R*.

Solution:

For each integer *a*,

$$[a] = \{x \in \mathbf{Z} \mid x \; R \; a\}$$

$$= \{x \in \mathbf{Z} \mid 3 \mid (x - a)\}$$

# Example 10 – *Solution*

cont'd

$$= \{x \in \mathbf{Z} \mid x - a = 3k, \text{ for some integer } k\}.$$

Therefore,

$$[a] = \{x \in \mathbf{Z} \mid x = 3k + a, \text{ for some integer } k\}.$$

In particular, $[0] = \{x \in \mathbf{Z} \mid x = 3k + 0, \text{ for some integer } k\}$

$$= \{x \in \mathbf{Z} \mid x = 3k, \text{ for some integer } k\}$$

$$= \{\ldots -9, -6, -3, 0, 3, 6, 9, \ldots\},$$

$$[1] = \{x \in \mathbf{Z} \mid x = 3k + 1, \text{ for some integer } k\}$$

$$= \{\ldots -8, -5, -2, 1, 4, 7, 10, \ldots\},$$

# Example 10 – *Solution*

cont'd

$$[2] = \{x \in \mathbf{Z} \mid x = 3k + 2, \text{ for some integer } k\}$$

$$= \{\ldots -7, -4, -1, 2, 5, 8, 11, \ldots\}.$$

Now since 3 *R* 0, then by Lemma 8.3.2,

$$[3] = [0].$$

More generally, by the same reasoning,

$$[0] = [3] = [-3] = [6] = [-6] = \ldots, \text{ and so on.}$$

Similarly,

$$[1] = [4] = [-2] = [7] = [-5] = \ldots, \text{ and so on.}$$

# Example 10 – *Solution*

cont'd

And

$$[2] = [5] = [-1] = [8] = [-4] = \ldots, \text{ and so on.}$$

Notice that every integer is in class [0], [1], or [2]. Hence the distinct equivalence classes are

$$\{x \in \mathbf{Z} \mid x = 3k, \text{ for some integer } k\},$$

$$\{x \in \mathbf{Z} \mid x = 3k + 1, \text{ for some integer } k\}, \quad \text{and}$$

$$\{x \in \mathbf{Z} \mid x = 3k + 2, \text{ for some integer } k\}.$$

# Example 10 – *Solution*

cont'd

In words, the three classes of congruence modulo 3 are (1) the set of all integers that are divisible by 3, (2) the set of all integers that leave a remainder of 1 when divided by 3, and (3) the set of all integers that leave a remainder of 2 when divided by 3.

# Congruence Modulo $n$

**• Definition**

Suppose $R$ is an equivalence relation on a set $A$ and $S$ is an equivalence class of $R$. $A$ **representative** of the class $S$ is any element $a$ such that $[a] = S$.

**• Definition**

Let $m$ and $n$ be integers and let $d$ be a positive integer. We say that $\boldsymbol{m}$ **is congruent to** $\boldsymbol{n}$ **modulo** $\boldsymbol{d}$ and write

$$\boldsymbol{m \equiv n \pmod{d}}$$

if, and only if, $$d \mid (m - n).$$

Symbolically: $$m \equiv n \pmod{d} \quad \Leftrightarrow \quad d \mid (m - n)$$

# Example 11 – *Evaluating Congruences*

Determine which of the following congruences are true and which are false.

**a.** $12 \equiv 7 \pmod 5$ $\quad$ $6 \equiv -8 \pmod 4$ $\quad$ $3 \equiv 3 \pmod 7$

Solution:

**a.** True. $12 - 7 = 5 = 5 \cdot 1$. Hence $5 \mid (12 - 7)$, and so $12 \equiv 7 \pmod 5$.

**b.** False. $6 - (-8) = 14$, and $4 \nmid 14$ because $14 \neq 4 \cdot k$ for any integer $k$. Consequently, $6 \not\equiv -8 \pmod 4$.

**c.** True. $3 - 3 = 0 = 7 \cdot 0$. Hence $7 \mid (3 - 3)$, and so $3 \equiv 3 \pmod 7$.

# A Definition for Rational Numbers

# A Definition for Rational Numbers

For a moment, forget what you know about fractional arithmetic and look at the numbers

$$\frac{1}{3} \quad \text{and} \quad \frac{2}{6}$$

as *symbols*. Considered as symbolic expressions, these *appear* quite different. In fact, if they were written as ordered pairs

$$(1, 3) \text{ and } (2, 6)$$

they would *be* different.

# A Definition for Rational Numbers

The fact that we regard them as “the same” is a specific instance of our general agreement to regard any two numbers

$$\frac{a}{b} \quad \text{and} \quad \frac{c}{d}$$

as equal provided the *cross products* are equal: $ad = bc$. This can be formalized as follows, using the language of equivalence relations.

## Example 12 – *Rational Numbers Are Really Equivalence Classes*

Let *A* be the set of all ordered pairs of integers for which the second element of the pair is nonzero. Symbolically,

$$A = \mathbf{Z} \times (\mathbf{Z} - \{0\}).$$

Define a relation *R* on *A* as follows: For all $(a, b), (c, d) \in A$,

$$(a, b)\ R\ (c, d) \quad \Leftrightarrow \quad ad = bc.$$

The fact is that *R* is an equivalence relation.

**a.** Prove that *R* is transitive.

**b.** Describe the distinct equivalence classes of *R*.

# Example 12(a) – *Solution*

Suppose ($a$, $b$), ($c$, d), and ($e$, $f$) are particular but arbitrarily chosen elements of $A$ such that ($a$, $b$) $R$ ($c$, $d$) and ($c$, $d$) $R$ ($e$, $f$).

*[We must show that for all* ($a$, $b$), ($c$, $d$), ($e$, $f$) $\in$ $A$, *if* ($a$, $b$) $R$ ($c$, $d$) *and* ($c$, $d$) $R$ ($e$, $f$), *then* ($a$, $b$) $R$ ($e$, $f$)*.]*

*[We must show that* ($a$, $b$) $R$ ($e$, $f$)*.]* By definition of $R$,

$$(1)\ \ ad = bc \quad \text{and} \quad (2)\ \ cf = de.$$

Since the second elements of all ordered pairs in $A$ are nonzero, $b \neq 0$, $d \neq 0$, and $f \neq 0$.

# Example 12(a) – *Solution*

cont'd

Multiply both sides of equation (1) by *f* and both sides of equation (2) by *b* to obtain

$$(1')\ \ adf = bcf \quad \text{and} \quad (2')\ \ bcf = bde.$$

Thus

$$adf = bde$$

and, since $d \neq 0$, it follows from the cancellation law for multiplication that

$$af = be.$$

It follows, by definition of *R*, that (*a*, *b*) *R* (*e*, *f*) *[as was to be shown].*

# Example 12(b) – *Solution*

cont'd

There is one equivalence class for each distinct rational number.

Each equivalence class consists of all ordered pairs ($a$, $b$) that, if written as fractions $a/b$, would equal each other.

The reason for this is that the condition for two rational numbers to be equal is the same as the condition for two ordered pairs to be related.

# Example 12(b) – *Solution*

cont'd

For instance, the class of (1, 2) is

$$[(1, 2)] = \{(1, 2), (-1, -2), (2, 4), (-2, -4), (3, 6), (-3, -6), \ldots\}$$

since $\frac{1}{2} = \frac{-1}{-2} = \frac{2}{4} = \frac{-2}{-4} = \frac{3}{6} = \frac{-3}{-6}$ and so forth.