CHAPTER 6

SET THEORY

SECTION 6.2

Properties of Sets

Properties of Sets

We begin by listing some set properties that involve subset relations.

Theorem 6.2.1 Some Subset Relations

1. *Inclusion of Intersection:* For all sets A and B,

$$(a)\ A \cap B \subseteq A \quad \text{and} \quad (b)\ A \cap B \subseteq B.$$

2. *Inclusion in Union:* For all sets A and B,

$$(a)\ A \subseteq A \cup B \quad \text{and} \quad (b)\ B \subseteq A \cup B.$$

3. *Transitive Property of Subsets:* For all sets A, B, and C,

$$\text{if } A \subseteq B \text{ and } B \subseteq C, \text{ then } A \subseteq C.$$

Properties of Sets

Procedural versions of the definitions of the other set operations are derived similarly and are summarized below.

Procedural Versions of Set Definitions

Let X and Y be subsets of a universal set U and suppose x and y are elements of U.

1. $x \in X \cup Y \quad \Leftrightarrow \quad x \in X$ or $x \in Y$
2. $x \in X \cap Y \quad \Leftrightarrow \quad x \in X$ and $x \in Y$
3. $x \in X - Y \quad \Leftrightarrow \quad x \in X$ and $x \notin Y$
4. $x \in X^c \quad \Leftrightarrow \quad x \notin X$
5. $(x, y) \in X \times Y \quad \Leftrightarrow \quad x \in X$ and $y \in Y$

Example 1 – *Proof of a Subset Relation*

Prove Theorem 6.2.1(1)(a): For all sets *A* and *B*, $A \cap B \subseteq A$.

Solution:

We start by giving a proof of the statement and then explain how you can obtain such a proof yourself.

Example 1 – *Solution*

cont'd

Proof:

Suppose A and B are any sets and suppose x is any element of $A \cap B$.

Then $x \in A$ and $x \in B$ by definition of intersection.

In particular, $x \in A$.

Thus $A \cap B \subseteq A$.

Example 1 – *Solution*

cont'd

The underlying structure of this proof is not difficult, but it is more complicated than the brevity of the proof suggests.

The first important thing to realize is that the statement to be proved is universal (it says that for *all* sets A and B, $A \cap B \subseteq A$).

The proof, therefore, has the following outline:

Starting Point: Suppose A and B are any (particular but arbitrarily chosen) sets.

To Show: $A \cap B \subseteq A$

Example 1 – *Solution*

cont'd

Now to prove that $A \cap B \subseteq A$, you must show that

$$\forall x, \text{ if } x \in A \cap B \text{ then } x \in A.$$

But this statement also is universal. So to prove it, you

suppose x is an element in $A \cap B$

and then you

show that x is in A.

Example 1 – *Solution*

cont'd

Filling in the gap between the "suppose" and the "show" is easy if you use the procedural version of the definition of intersection: To say that x is in $A \cap B$ means that

$$x \text{ is in } A \quad \text{and} \quad x \text{ is in } B.$$

This allows you to complete the proof by deducing that, in particular,

$$x \text{ is in } A,$$

as was to be shown.

Note that this deduction is just a special case of the valid argument form

$$p \wedge q$$

$$\bullet\ p.$$

Set Identities

Set Identities

An **identity** is an equation that is universally true for all elements in some set. For example, the equation $a + b = b + a$ is an identity for real numbers because it is true for all real numbers a and b.

The collection of set properties in the next theorem consists entirely of set identities. That is, they are equations that are true for all sets in some universal set.

Theorem 6.2.2 Set Identities

Let all sets referred to below be subsets of a universal set U.

1. *Commutative Laws:* For all sets A and B,

$$\text{(a) } A \cup B = B \cup A \quad \text{and} \quad \text{(b) } A \cap B = B \cap A.$$

Set Identities

cont'd

2. *Associative Laws:* For all sets A, B, and C,

$$\text{(a) } (A \cup B) \cup C = A \cup (B \cup C) \quad \text{and}$$
$$\text{(b) } (A \cap B) \cap C = A \cap (B \cap C).$$

3. *Distributive Laws:* For all sets, A, B, and C,

$$\text{(a) } A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \quad \text{and}$$
$$\text{(b) } A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

4. *Identity Laws:* For all sets A,

$$\text{(a) } A \cup \emptyset = A \quad \text{and} \quad \text{(b) } A \cap U = A.$$

5. *Complement Laws:*

$$\text{(a) } A \cup A^c = U \quad \text{and} \quad \text{(b) } A \cap A^c = \emptyset.$$

6. *Double Complement Law:* For all sets A,

$$(A^c)^c = A.$$

7. *Idempotent Laws:* For all sets A,

$$\text{(a) } A \cup A = A \quad \text{and} \quad \text{(b) } A \cap A = A.$$

Set Identities

cont'd

8. *Universal Bound Laws:* For all sets A,

$$\text{(a) } A \cup U = U \quad \text{and} \quad \text{(b) } A \cap \emptyset = \emptyset.$$

9. *De Morgan's Laws:* For all sets A and B,

$$\text{(a) } (A \cup B)^c = A^c \cap B^c \quad \text{and} \quad \text{(b) } (A \cap B)^c = A^c \cup B^c.$$

10. *Absorption Laws:* For all sets A and B,

$$\text{(a) } A \cup (A \cap B) = A \quad \text{and} \quad \text{(b) } A \cap (A \cup B) = A.$$

11. *Complements of U and $\emptyset$:*

$$\text{(a) } U^c = \emptyset \quad \text{and} \quad \text{(b) } \emptyset^c = U.$$

12. *Set Difference Law:* For all sets A and B,

$$A - B = A \cap B^c.$$

Proving Set Identities

Proving Set Identities

As we have known,

Two sets are equal ⇔ each is a subset of the other.

The method derived from this fact is the most basic way to prove equality of sets.

Basic Method for Proving That Sets Are Equal

Let sets X and Y be given. To prove that $X = Y$:

1. Prove that $X \subseteq Y$.
2. Prove that $Y \subseteq X$.

Example 2 – *Proof of a Distributive Law*

Prove that for all sets A, B, and C,

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

Solution:

The proof of this fact is somewhat more complicated than the proof in Example 1, so we first derive its logical structure, then find the core arguments, and end with a formal proof as a summary.

Example 2 – *Solution*

cont'd

As in Example 1, the statement to be proved is universal, and so, by the method of generalizing from the generic particular, the proof has the following outline:

Starting Point: Suppose A, B, and C are arbitrarily chosen sets.

To Show: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Example 2 – *Solution*

cont'd

Now two sets are equal if, and only if, each is a subset of the other.

Hence, the following two statements must be proved:

$$A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$$

and $$(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C).$$

Showing the first containment requires showing that

$$\forall x, \text{ if } x \in A \cup (B \cap C) \text{ then } x \in (A \cup B) \cap (A \cup C).$$

Example 2 – *Solution*

cont'd

Showing the second containment requires showing that

$$\forall x, \text{ if } x \in (A \cup B) \cap (A \cup C) \text{ then } x \in A \cup (B \cap C).$$

Note that both of these statements are universal. So to prove the first containment, you

suppose you have any element x in $A \cup (B \cap C)$,

and then you

show that $x \in (A \cup B) \cap (A \cup C)$.

Example 2 – *Solution*

cont'd

And to prove the second containment, you

suppose you have any element *x* in $(A \cup B) \cap (A \cup C)$,

and then you

show that $x \in A \cup (B \cap C)$.

Example 2 – *Solution*

cont'd

In Figure 6.2.1, the structure of the proof is illustrated by the kind of diagram that is often used in connection with structured programs.

Suppose A, B, and C are sets. *[Show $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. That is, show $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$ and $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$.]*

> Show $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$. *[That is, show $\forall x$, if $x \in A \cup (B \cap C)$ then $x \in (A \cup B) \cap (A \cup C)$.]*
>
> > Suppose $x \in A \cup (B \cap C)$. *[Show $x \in (A \cup B) \cap (A \cup C)$.]*
> >
> > $\vdots$
> >
> > Thus $x \in (A \cup B) \cap (A \cup C)$.
>
> Hence $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$.

Figure 6.2.1

Example 2 – *Solution*

cont'd

Show $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$. *[That is, show $\forall x$, if $x \in (A \cup B) \cap (A \cup C)$ then $x \in A \cup (B \cap C)$.]*

Suppose $x \in (A \cup B) \cap (A \cup C)$. *[Show $x \in A \cup (B \cap C)$.]*

$\vdots$

Thus $x \in A \cup (B \cap C)$.

Hence $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$.

Thus $(A \cup B) \cap (A \cup C) = A \cup (B \cap C)$.

Figure 6.2.1 (continued)

The analysis in the diagram reduces the proof to two concrete tasks: filling in the steps indicated by dots in the two center boxes of Figure 6.2.1.

Example 2 – *Solution*

cont'd

Filling in the missing steps in the top box:
To fill in these steps, you go from the supposition that $x \in A \cup (B \cap C)$ to the conclusion that $x \in (A \cup B) \cap (A \cup C)$.

Now when $x \in A \cup (B \cap C)$, then by definition of union, $x \in A$ or $x \in B \cap C$. But either of these possibilities might be the case because x is assumed to be chosen arbitrarily from the set $A \cup (B \cap C)$.

So you have to show you can reach the conclusion that $x \in (A \cup B) \cap (A \cup C)$ regardless of whether x happens to be in A or x happens to be in $B \cap C$.

Example 2 – *Solution*

cont'd

This leads you to break your analysis into two cases: $x \in A$ and $x \in B \cap C$.

In case $x \in A$, your goal is to show that $x \in (A \cup B) \cap (A \cup C)$, which means that $x \in A \cup B$ and $x \in A \cup C$ (by definition of intersection). But when $x \in$ A, both statements $x \in A \cup B$ and $x \in A \cup C$ are true by virtue of x's being in A.

Similarly, in case $x \in B \cap C$, your goal is also to show that $x \in (A \cup B) \cap (A \cup C)$, which means that $x \in A \cup B$ and $x \in A \cup C$.

Example 2 – *Solution*

cont'd

But when $x \in B \cap C$, then $x \in B$ and $x \in C$ (by definition of intersection), and so $x \in A \cup B$ (by virtue of being in B) and $x \in A \cup C$ (by virtue of being in C).

This analysis shows that regardless of whether $x \in A$ or $x \in B \cap C$, the conclusion $x \in (A \cup B) \cap (A \cup C)$ follows. So you can fill in the steps in the top inner box.

Filling in the missing steps in the bottom box:

To fill in these steps, you need to go from the supposition that $x \in (A \cup B) \cap (A \cup C)$ to the conclusion that $x \in A \cup (B \cap C)$.

Example 2 – *Solution*

cont'd

When $x \in (A \cup B) \cap (A \cup C)$ it is natural to consider the two cases $x \in A$ and $x \notin A$ because when x happens to be in A, then the statement "$x \in A$ or $x \in B \cap C$" is certainly true, and so x is in $A \cup (B \cap C)$ by definition of union.

Thus it remains only to show that even in the case when x is not in A, and $x \in (A \cup B) \cap (A \cup C)$, then $x \in A \cup (B \cap C)$.

So suppose x is not in A. Now to say that $x \in (A \cup B) \cap (A \cup C)$ means that $x \in A \cup B$ and $x \in A \cup C$ (by definition of intersection). But when $x \in A \cup B$, then x is in at least one of A or B, so since x is not in A, then x must be in B.

Example 2 – *Solution*

cont'd

Similarly, when $x \in A \cup C$, then x is in at least one of A or C, so since x is not in A, then x must be in C. Thus, when x is not in A and $x \in (A \cup B) \cap (A \cup C)$, then x is in both B and C, which means that $x \in B \cap C$.

It follows that the statement "$x \in A$ or $x \in B \cap C$" is true, and so $x \in A \cup (B \cap C)$ by definition of union.

This analysis shows that if $x \in (A \cup B) \cap (A \cup C)$, then regardless of whether $x \in A$ or $x \notin A$, you can conclude that $x \in A \cup (B \cap C)$. Hence you can fill in the steps of the bottom inner box.

Proving Set Identities

Suppose *A* and *B* are arbitrarily chosen sets.

Theorem 6.2.2(3)(a) A Distributive Law for Sets

For all sets A, B, and C,

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

Theorem 6.2.2(9)(a) A De Morgan's Law for Sets

For all sets A and B, $(A \cup B)^c = A^c \cap B^c$.

Proving Set Identities

The set property given in the next theorem says that if one set is a subset of another, then their intersection is the smaller of the two sets and their union is the larger of the two sets.

Theorem 6.2.3 Intersection and Union with a Subset

For any sets A and B, if $A \subseteq B$, then

$$\text{(a) } A \cap B = A \quad \text{and} \quad \text{(b) } A \cup B = B.$$

The Empty Set

The Empty Set

The crucial fact is that the negation of a universal statement is existential: If a set *B* is not a subset of a set *A*, then there exists an element in *B* that is not in *A*. But if *B* has no elements, then no such element can exist.

Theorem 6.2.4 A Set with No Elements Is a Subset of Every Set

If E is a set with no elements and A is any set, then $E \subseteq A$.

If *E* is a set with no elements and *A* is any set, then to say that $E \subseteq A$ is the same as saying that

$$\forall x, \text{ if } x \in E, \text{ then } x \in A.$$

The Empty Set

But since *E* has no elements, this conditional statement is vacuously true.

How many sets with no elements are there? Only one.

> **Corollary 6.2.5 Uniqueness of the Empty Set**
>
> There is only one set with no elements.

Suppose you need to show that a certain set equals the empty set. By Corollary 6.2.5 it suffices to show that the set has no elements.

The Empty Set

For since there is only one set with no elements (namely ∅), if the given set has no elements, then it must equal ∅.

Element Method for Proving a Set Equals the Empty Set

To prove that a set X is equal to the empty set $\emptyset$, prove that X has no elements. To do this, suppose X has an element and derive a contradiction.

Example 5 – *A Proof for a Conditional Statement*

Prove that for all sets A, B, and C, if $A \subseteq B$ and $B \subseteq C^c$, then $A \cap C = \varnothing$.

Solution:

Since the statement to be proved is both universal and conditional, you start with the method of direct proof:

Suppose A, B, and C are arbitrarily chosen sets that satisfy the condition: $A \subseteq B$ and $B \subseteq C^c$.

Show that $A \cap C = \varnothing$.

Example 5 – *Solution*

cont'd

Since the conclusion to be shown is that a certain set is empty, you can use the principle for proving that a set equals the empty set.

A complete proof is shown below.

> **Proposition 6.2.6**
>
> For all sets A, B, and C, if $A \subseteq B$ and $B \subseteq C^c$, then $A \cap C = \emptyset$.

Example 5 – *Solution*

cont'd

Proof:

Suppose A, B, and C are any sets such that $A \subseteq B$ and $B \subseteq C^c$. We must show that $A \cap C = \varnothing$. Suppose not. That is, suppose there is an element x in $A \cap C$.

By definition of intersection, $x \in A$ and $x \in C$. Then, since $A \subseteq B$, $x \in B$ by definition of subset. Also, since $B \subseteq C^c$, then $x \in C^c$ by definition of subset again. It follows by definition of complement that $x \notin C$. Thus $x \in C$ and $x \notin C$, which is a contradiction.

So the supposition that there is an element x in $A \cap C$ is false, and thus $A \cap C = \varnothing$ *[as was to be shown].*