CHAPTER 6

SET THEORY

SECTION 6.1

Set Theory: Definitions and the Element Method of Proof

Set Theory: Definitions and the Element Method of Proof

The words *set* and *element* are undefined terms of set theory just as *sentence*, *true*, and *false* are undefined terms of logic.

The founder of set theory, Georg Cantor, suggested imagining a set as a "collection into a whole *M* of definite and separate objects of our intuition or our thought. These objects are called the elements of *M*."

Cantor used the letter *M* because it is the first letter of the German word for set: *Menge*.

Subsets: Proof and Disproof

Subsets: Proof and Disproof

We begin by rewriting what it means for a set *A* to be a subset of a set *B* as a formal universal conditional statement:

$$A \subseteq B \quad \Leftrightarrow \quad \forall x, \text{ if } x \in A \text{ then } x \in B.$$

The negation is, therefore, existential:

$$\mathrm{A} \nsubseteq \mathrm{B} \quad \Leftrightarrow \quad \exists x \text{ such that } x \in A \text{ and } x \notin B.$$

Subsets: Proof and Disproof

A *proper subset* of a set is a subset that is not equal to its containing set. Thus

> A is a **proper subset** of B $\Leftrightarrow$
>
> (1) $A \subseteq B$, and
>
> (2) there is at least one element in B that is not in A.

Example 1 – *Testing Whether One Set Is a Subset of Another*

Let $A = \{1\}$ and $B = \{1, \{1\}\}$.

a. Is $A \subseteq B$?

b. If so, is A a proper subset of B?

Solution:

a. Because $A = \{1\}$, A has only one element, namely the symbol 1.

This element is also one of the elements in set B. Hence every element in A is in B, and so $A \subseteq B$.

Example 1 – *Solution*

cont'd

b. B has two distinct elements, the symbol 1 and the set {1} whose only element is 1.

Since $1 \neq \{1\}$, the set {1} is not an element of A, and so there is an element of B that is not an element of A. Hence A is a proper subset of B.

Subsets: Proof and Disproof

Because we define what it means for one set to be a subset of another by means of a universal conditional statement, we can use the method of direct proof to establish a subset relationship.

Such a proof is called an *element argument* and is the fundamental proof technique of set theory.

Element Argument: The Basic Method for Proving That One Set Is a Subset of Another

Let sets X and Y be given. To prove that $X \subseteq Y$,

1. **suppose** that x is a particular but arbitrarily chosen element of X,
2. **show** that x is an element of Y.

Example 2 – *Proving and Disproving Subset Relations*

Define sets *A* and *B* as follows:

$$A = \{m \in \mathbf{Z} \mid m = 6r + 12 \text{ for some } r \in \mathbf{Z}\}$$
$$B = \{n \in \mathbf{Z} \mid n = 3s \text{ for some } s \in \mathbf{Z}\}.$$

a. Outline a proof that $A \subseteq B$.

b. Prove that $A \subseteq B$.

c. Disprove that $B \subseteq A$.

Example 2 – *Solution*

a. Proof Outline:

Suppose x is a particular but arbitrarily chosen element of A.

.

.

.

Therefore, x is an element of B.

b. Proof:

Suppose x is a particular but arbitrarily chosen element of A.

[We must show that $x \in B$. By definition of B, this means we must show that $x = 3 \cdot$ (some integer).]

Example 2 – *Solution*

cont'd

By definition of A, there is an integer r such that $x = 6r + 12$.

[Given that $x = 6r + 12$, *can we express* x *as* $3 \cdot$(*some integer*)?
I.e., does $6r + 12 = 3 \cdot$(*some integer*)? *Yes,* $6r + 12 = 3 \cdot (2r + 4)$.*]*

Let $s = 2r + 4$.

[We must check that s is an integer.]

Then s is an integer because products and sums of integers are integers.

[Now we must check that $x = 3s$.*]*

Example 2 – *Solution*

cont'd

Also $3s = 3(2r + 4) = 6r + 12 = x$,

Thus, by definition of *B*, *x* is an element of *B*,

[which is what was to be shown].

c. To disprove a statement means to show that it is false, and to show it is false that $B \subseteq A$, you must find an element of *B* that is not an element of *A*.

Example 2 – *Solution*

cont'd

By the definitions of A and B, this means that you must find an integer x of the form $3 \cdot$ (some integer) that cannot be written in the form $6 \cdot$ (some integer) $+ 12$.

A little experimentation reveals that various numbers do the job. For instance, you could let $x = 3$.

Then $x \in B$ because $3 = 3 \cdot 1$, but $x \notin A$ because there is no integer r such that $3 = 6r + 12$. For if there were such an integer, then

$$6r + 12 = 3 \qquad \text{by assumption}$$

Example 2 – *Solution*

cont'd

$$\Rightarrow \quad 2r + 4 = 1 \quad \text{by dividing both sides by 3}$$

$$\Rightarrow \quad 2r = 3 \quad \text{by subtracting 4 from both sides}$$

$$\Rightarrow \quad r = 3/2 \quad \text{by dividing both sides by 2,}$$

but 3/2 is not an integer. Thus $3 \in B$ but $3 \notin A$, and so $B \nsubseteq A$.

Set Equality

Set Equality

We have known that by the axiom of extension, sets *A* and *B* are equal if, and only if, they have exactly the same elements.

We restate this as a definition that uses the language of subsets.

• Definition

Given sets A and B, A **equals** B, written $\boldsymbol{A = B}$, if, and only if, every element of A is in B and every element of B is in A.

Symbolically:

$$A = B \quad \Leftrightarrow \quad A \subseteq B \text{ and } B \subseteq A.$$

Set Equality

This version of the definition of equality implies the following:

> To know that a set A equals a set B, you must know that $A \subseteq B$ and you must also know that $B \subseteq A$.

Example 3 – *Set Equality*

Define sets *A* and *B* as follows:

$$A = \{m \in \mathbf{Z} \mid m = 2a \text{ for some integer } a\}$$
$$B = \{n \in \mathbf{Z} \mid n = 2b - 2 \text{ for some integer } b\}$$

Is $A = B$?

Solution:

Yes. To prove this, both subset relations $A \subseteq B$ and $B \subseteq A$ must be proved.

Example 3 – *Solution*

cont'd

Part 1, Proof That $A \subseteq B$:

Suppose x is a particular but arbitrarily chosen element of A.

[We must show that $x \in B$. By definition of B, this means we must show that $x = 2 \cdot$ (some integer) $- 2$.]

By definition of A, there is an integer a such that $x = 2a$.

[Given that $x = 2a$, can x also be expressed as $2 \cdot$ (some integer) $- 2$? i.e., is there an integer, say b, such that $2a = 2b - 2$? Solve for b to obtain $b = (2a + 2)/2 = a + 1$. Check to see if this works.]

Example 3 – *Solution*

cont'd

Let $b = a + 1$.

[First check that b is an integer.]

Then b is an integer because it is a sum of integers.

[Then check that $x = 2b - 2$.]

Also $2b - 2 = 2(a + 1) - 2 = 2a + 2 - 2 = 2a = x$,

Thus, by definition of B, x is an element of B

[which is what was to be shown].

Part 2, Proof That $B \subseteq A$:

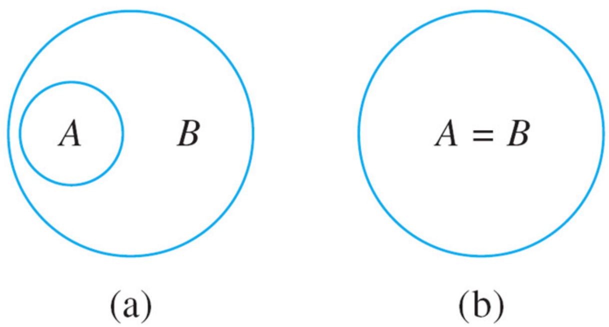
Similarly we can prove that $B \subseteq A$. Hence $A = B$.

Venn Diagrams

Venn Diagrams

If sets *A* and *B* are represented as regions in the plane, relationships between *A* and *B* can be represented by pictures, called **Venn diagrams,** that were introduced by the British mathematician John Venn in 1881.

For instance, the relationship $A \subseteq B$ can be pictured in one of two ways, as shown in Figure 6.1.1.

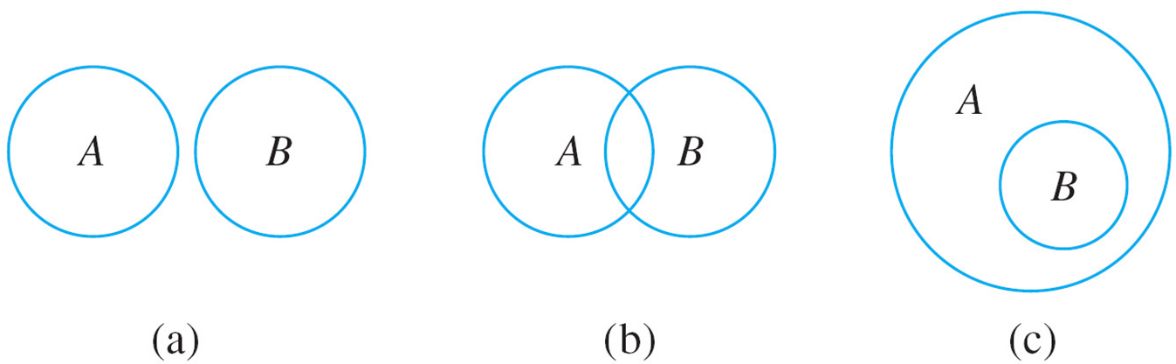


$A \subseteq B$

Figure 6.1.1

Venn Diagrams

The relationship $A \not\subseteq B$ can be represented in three different ways with Venn diagrams, as shown in Figure 6.1.2.



$A \not\subseteq B$

Figure 6.1.2

Example 4 – *Relations among Sets of Numbers*

Since **Z**, **Q**, and **R** denote the sets of integers, rational numbers, and real numbers, respectively, **Z** is a subset of **Q** because every integer is rational (any integer *n* can be written in the form $\frac{n}{1}$).

Q is a subset of **R** because every rational number is real (any rational number can be represented as a length on the number line).

Z is a proper subset of **Q** because there are rational numbers that are not integers (for example, $\frac{1}{2}$).

Example 4 – *Relations among Sets of Numbers*

cont'd

Q is a proper subset of **R** because there are real numbers that are not rational (for example, $\sqrt{2}$).

This is shown diagrammatically in Figure 6.1.3.

Figure 6.1.3

Operations on Sets

Operations on Sets

Most mathematical discussions are carried on within some context. For example, in a certain situation all sets being considered might be sets of real numbers.

In such a situation, the set of real numbers would be called a **universal set** or a **universe of discourse** for the discussion.

Operations on Sets

• Definition

Let A and B be subsets of a universal set U.

1. The **union** of A and B, denoted $\boldsymbol{A \cup B}$**,** is the set of all elements that are in at least one of A or B.

2. The **intersection** of A and B, denoted $\boldsymbol{A \cap B}$**,** is the set of all elements that are common to both A and B.

3. The **difference** of B minus A (or **relative complement** of A in B), denoted $\boldsymbol{B - A}$**,** is the set of all elements that are in B and not A.

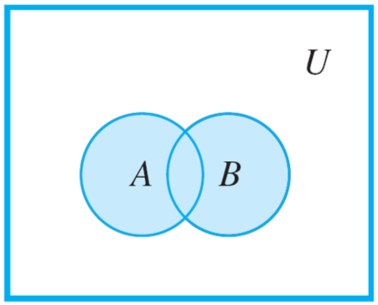
4. The **complement** of A, denoted $\boldsymbol{A^c}$**,** is the set of all elements in U that are not in A.

Symbolically:

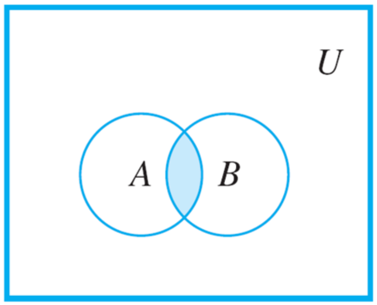
$$A \cup B = \{x \in U \mid x \in A \text{ or } x \in B\},$$
$$A \cap B = \{x \in U \mid x \in A \text{ and } x \in B\},$$
$$B - A = \{x \in U \mid x \in B \text{ and } x \notin A\},$$
$$A^c = \{x \in U \mid x \notin A\}.$$

Operations on Sets

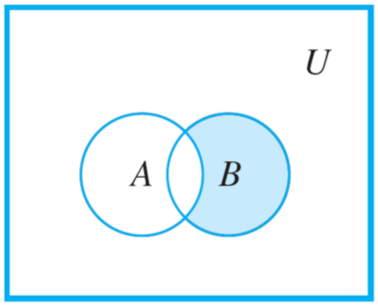
Venn diagram representations for union, intersection, difference, and complement are shown in Figure 6.1.4.



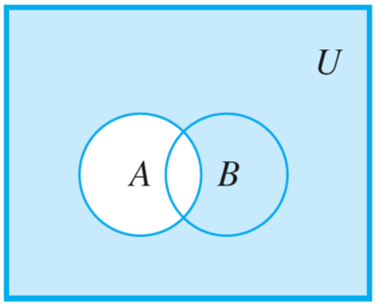
Shaded region represents $A \cup B$.



Shaded region represents $A \cap B$.



Shaded region represents $B - A$.



Shaded region represents A^c.

Figure 6.1.4

Example 5 – *Unions, Intersections, Differences, and Complements*

Let the universal set be the set $U = \{a, b, c, d, e, f, g\}$ and let $A = \{a, c, e, g\}$ and $B = \{d, e, f, g\}$. Find $A \cup B$, $A \cap B$, $B - A$, and A^c.

Solution:

$$A \cup B = \{a, c, d, e, f, g\}$$

$$A \cap B = \{e, g\}$$

$$B - A = \{d, f\}$$

$$A^c = \{b, d, f\}$$

Operations on Sets

There is a convenient notation for subsets of real numbers that are intervals.

• Notation

Given real numbers a and b with $a \leq b$:

$$(a, b) = \{x \in \mathbf{R} \mid a < x < b\} \qquad [a, b] = \{x \in \mathbf{R} \mid a \leq x \leq b\}$$
$$(a, b] = \{x \in \mathbf{R} \mid a < x \leq b\} \qquad [a, b) = \{x \in \mathbf{R} \mid a \leq x < b\}.$$

The symbols ∞ and $-\infty$ are used to indicate intervals that are unbounded either on the right or on the left:

$$(a, \infty) = \{x \in \mathbf{R} \mid x > a\} \qquad [a, \infty) = \{x \in \mathbf{R} \mid x \geq a\}$$
$$(-\infty, b) = \{x \in \mathbf{R} \mid x < b\} \qquad [-\infty, b) = \{x \in \mathbf{R} \mid x \leq b\}.$$

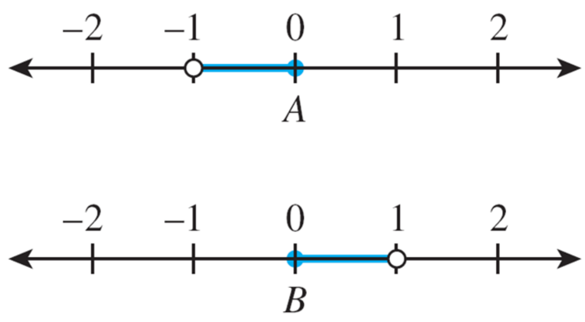
Observe that the notation for the interval (a, b) is identical to the notation for the ordered pair (a, b). However, context makes it unlikely that the two will be confused.

Example 6 – *An Example with Intervals*

Let the universal set be the set **R** of all real numbers and let

$$A = (-1,\ 0] = \{x \in \mathbf{R} \mid -1 < x \le 0\} \text{ and } B = [0,\ 1) = \{x \in \mathbf{R} \mid 0 \le x < 1\}.$$

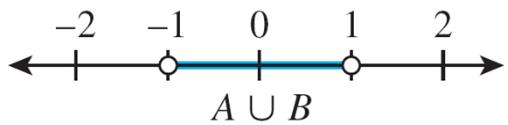
These sets are shown on the number lines below.



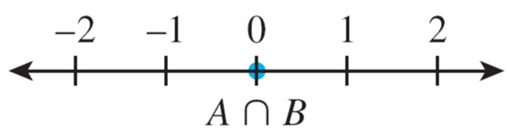
Find $A \cup B$, $A \cap B$, $B - A$, and A^c.

Example 6 – *Solution*

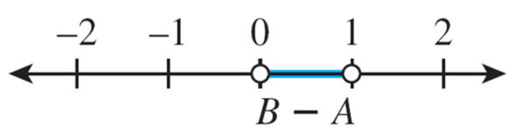
$$A \cup B = \{x \in \mathbf{R} \mid x \in (-1,\ 0] \text{ or } x \in [0,\ 1)\} = \{x \in \mathbf{R} \mid x \in (-1,\ 1)\} = (-1,\ 1).$$



$$A \cap B = \{x \in \mathbf{R} \mid x \in (-1,\ 0] \text{ and } x \in [0,\ 1)\} = \{0\}.$$



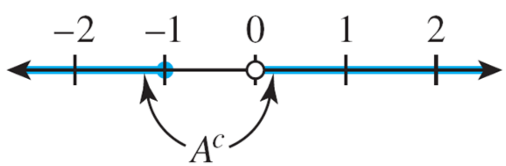
$$B - A = \{x \in \mathbf{R} \mid x \in [0,\ 1) \text{ and } x \notin (-1,\ 0]\} = \{x \in \mathbf{R} \mid 0 < x < 1\} = (0, 1)$$



Example 6 – *Solution*

cont'd

$$A^c = \{x \in \mathbf{R} \mid \text{it is not the case that } x \in (-1,\ 0]\}$$



$$= \{x \in \mathbf{R} \mid \text{it is not the case that } (-1 < x \text{ and } x \leq 0)\}$$

by definition of the double inequality

$$= \{x \in \mathbf{R} \mid x \leq -1 \text{ or } x > 0\} = (-\infty,\ -1] \cup (0,\ \infty)$$

by De Morgan's law

Operations on Sets

The definitions of unions and intersections for more than two sets are very similar to the definitions for two sets.

• Definition

Unions and Intersections of an Indexed Collection of Sets

Given sets $A_0, A_1, A_2, \ldots$ that are subsets of a universal set U and given a nonnegative integer n,

$$\bigcup_{i=0}^{n} A_i = \{x \in U \mid x \in A_i \text{ for at least one } i = 0, 1, 2, \ldots, n\}$$

$$\bigcup_{i=0}^{\infty} A_i = \{x \in U \mid x \in A_i \text{ for at least one nonnegative integer } i\}$$

$$\bigcap_{i=0}^{n} A_i = \{x \in U \mid x \in A_i \text{ for all } i = 0, 1, 2, \ldots, n\}$$

$$\bigcap_{i=0}^{\infty} A_i = \{x \in U \mid x \in A_i \text{ for all nonnegative integers } i\}.$$

Operations on Sets

An alternative notation for $\bigcup_{i=0}^{n} A_i$ is $A_0 \cup A_1 \cup \ldots \cup A_n$, and an alternative notation for $\bigcap_{i=0}^{n} A_i$ is $A_0 \cap A_1 \cap \ldots \cap A_n$.

Example 7 – *Finding Unions and Intersections of More than Two Sets*

For each positive integer *i*, let

$$A_i = \left\{x \in \mathbf{R} \mid -\frac{1}{i} < x < \frac{1}{i}\right\} = A_i = \left(-\frac{1}{i}, \frac{1}{i}\right).$$

a. Find $A_1 \cup A_2 \cup A_3$ and $A_1 \cap A_2 \cap A_3$.

b. Find $\bigcup_{i=1}^{\infty} A_i$ and $\bigcap_{i=1}^{\infty} A_i$.

Solution:

a. $A_1 \cup A_2 \cup A_3 = \{x \in \mathbf{R} \mid x$ is in at least one of the intervals $(-1, 1)$,

or $\left(-\frac{1}{2}, \frac{1}{2}\right)$, or $\left(-\frac{1}{3}, \frac{1}{3}\right)\}$

Example 7 – *Solution*

cont'd

$$= \{x \in \mathbf{R} \mid -1 < x < 1\}$$ because all the elements in $\left(-\frac{1}{2}, \frac{1}{2}\right)$ and $\left(-\frac{1}{3}, \frac{1}{3}\right)$ are in $(-1, 1)$

$$= (-1, 1)$$

$$A_1 \cap A_2 \cap A_3 = \{x \in \mathbf{R} \mid x \text{ is in all of the intervals } (-1, 1), \text{ and } \left(-\tfrac{1}{2}, \tfrac{1}{2}\right), \text{ and } \left(-\tfrac{1}{3}, \tfrac{1}{3}\right)\}$$

$$= \left\{x \in \mathbf{R} \mid -\frac{1}{3} < x < \frac{1}{3}\right\}$$ because $\left(-\frac{1}{3}, \frac{1}{3}\right) \subseteq \left(-\frac{1}{2}, \frac{1}{2}\right) \subseteq (-1, 1)$

$$= \left(-\frac{1}{3}, \frac{1}{3}\right)$$

Example 7 – *Solution*

cont'd

b. $\bigcup_{i=1}^{\infty} A_i = \{x \in \mathbf{R} \mid x \text{ is in at least one of the intervals } \left(-\frac{1}{i}, \frac{1}{i}\right),$ where i is a positive integer$\}$

$= \{x \in \mathbf{R} \mid -1 < x < 1\}$ because all the elements in every interval $\left(-\frac{1}{i}, \frac{1}{i}\right)$ are in $(-1, 1)$

$= (-1, 1)$

$\bigcap_{i=1}^{\infty} A_i = \{x \in \mathbf{R} \mid x \text{ is in all of the intervals } \left(-\frac{1}{i}, \frac{1}{i}\right),$ where i is a positive integer$\}$

$= \{0\}$ because the only element in every interval is 0

The Empty Set

The Empty Set

We have stated that a set is defined by the elements that compose it. This being so, can there be a set that has no elements? It turns out that it is convenient to allow such a set.

Because it is unique, we can give it a special name. We call it the **empty set** (or **null set**) and denote it by the symbol Ø.

Thus $\{1, 3\} \cap \{2, 4\} = \varnothing$ and $\{x \in \mathbf{R} \mid x^2 = -1\} = \varnothing$.

Example 8 – *A Set with No Elements*

Describe the set $D = \{x \in \mathbf{R} \mid 3 < x < 2\}$.

Solution:

We have known that $a < x < b$ means that $a < x$ and $x < b$. So D consists of all real numbers that are both greater than 3 and less than 2.

Since there are no such numbers, D has no elements and so $D = \varnothing$.

Partitions of Sets

Partitions of Sets

In many applications of set theory, sets are divided up into nonoverlapping (or *disjoint*) pieces. Such a division is called a *partition*.

• **Definition**

Two sets are called **disjoint** if, and only if, they have no elements in common.
Symbolically:

$$A \text{ and } B \text{ are disjoint} \quad \Leftrightarrow \quad A \cap B = \emptyset.$$

Example 9 – *Disjoint Sets*

Let $A = \{1, 3, 5\}$ and $B = \{2, 4, 6\}$. Are A and B disjoint?

Solution:

Yes. By inspection A and B have no elements in common, or, in other words, $\{1, 3, 5\} \cap \{2, 4, 6\} = \varnothing$.

Partitions of Sets

- **Definition**

Sets $A_1, A_2, A_3 \ldots$ are **mutually disjoint** (or **pairwise disjoint** or **nonoverlapping**) if, and only if, no two sets A_i and A_j with distinct subscripts have any elements in common. More precisely, for all $i, j = 1, 2, 3, \ldots$

$$A_i \cap A_j = \emptyset \quad \text{whenever } i \neq j.$$

Example 10 – *Mutually Disjoint Sets*

a. Let $A_1 = \{3, 5\}$, $A_2 = \{1, 4, 6\}$, and $A_3 = \{2\}$. Are A_1, A_2, and A_3 mutually disjoint?

b. Let $B_1 = \{2, 4, 6\}$, $B_2 = \{3, 7\}$, and $B_3 = \{4, 5\}$. Are B_1, B_2, and B_3 mutually disjoint?

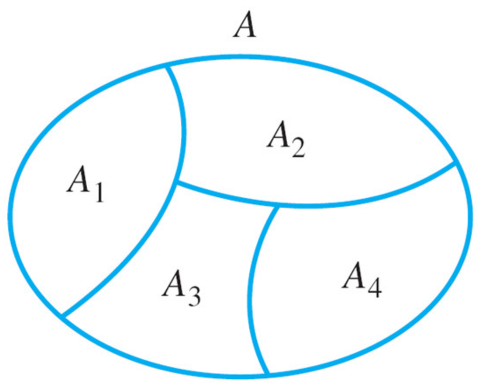
Solution:

a. Yes. A_1 and A_2 have no elements in common, A_1 and A_3 have no elements in common, and A_2 and A_3 have no elements in common.

b. No. B_1 and B_3 both contain 4.

Partitions of Sets

Suppose A, A_1, A_2, A_3, and A_4 are the sets of points represented by the regions shown in Figure 6.1.5.



A Partition of a Set

Figure 6.1.5

Then A_1, A_2, A_3, and A_4 are subsets of A, and

$A = A_1 \cup A_2 \cup A_3 \cup A_4$.

Partitions of Sets

Suppose further that boundaries are assigned to the regions representing A_2, A_3, and A_4 in such a way that these sets are mutually disjoint.

Then A is called a *union of mutually disjoint subsets*, and the collection of sets $\{A_1, A_2, A_3, A_4\}$ is said to be a *partition* of A.

• Definition

A finite or infinite collection of nonempty sets $\{A_1, A_2, A_3 \ldots\}$ is a **partition** of a set A if, and only if,

1. A is the union of all the A_i
2. The sets $A_1, A_2, A_3, \ldots$ are mutually disjoint.

Example 11 – *Partitions of Sets*

a. Let $A = \{1, 2, 3, 4, 5, 6\}$, $A_1 = \{1, 2\}$, $A_2 = \{3, 4\}$, and $A_3 = \{5, 6\}$. Is $\{A_1, A_2, A_3\}$ a partition of A?

b. Let **Z** be the set of all integers and let

$$T_0 = \{n \in \mathbf{Z} \mid n = 3k, \text{ for some integer } k\},$$
$$T_1 = \{n \in \mathbf{Z} \mid n = 3k + 1, \text{ for some integer } k\}, \text{ and}$$
$$T_2 = \{n \in \mathbf{Z} \mid n = 3k + 2, \text{ for some integer } k\}.$$

Is $\{T_0, T_1, T_2\}$ a partition of **Z**?

Example 11 – *Solution*

a. Yes. By inspection, $A = A_1 \cup A_2 \cup A_3$ and the sets A_1, A_2, and A_3 are mutually disjoint.

b. Yes. By the quotient-remainder theorem, every integer n can be represented in exactly one of the three forms

$$n = 3k \quad \text{or} \quad n = 3k + 1 \quad \text{or} \quad n = 3k + 2,$$

for some integer k.

This implies that no integer can be in any two of the sets T_0, T_1, or T_2. So T_0, T_1, and T_2 are mutually disjoint.

It also implies that every integer is in one of the sets T_0, T_1, or T_2. So $\mathbf{Z} = T_0 \cup T_1 \cup T_2$.

Power Sets

Power Sets

There are various situations in which it is useful to consider the set of all subsets of a particular set.

The **power set axiom** guarantees that this is a set.

• Definition

Given a set A, the **power set** of A, denoted $\mathscr{P}(A)$, is the set of all subsets of A.

Example 12 – *Power Set of a Set*

Find the power set of the set $\{x, y\}$. That is, find $\mathscr{P}(\{x, y\})$.

Solution:

$\mathscr{P}(\{x, y\})$ is the set of all subsets of $\{x, y\}$. We know that Ø is a subset of every set, and so $\emptyset \in \mathscr{P}(\{x, y\})$.

Also any set is a subset of itself, so $\{x, y\} \in \mathscr{P}(\{x, y\})$. The only other subsets of $\{x, y\}$ are $\{x\}$ and $\{y\}$, so

$$\mathscr{P}(\{x, y\}) = \{\emptyset, \{x\}, \{y\}, \{x, y\}\}.$$

Cartesian Products

Cartesian Products

• Definition

Let n be a positive integer and let $x_1, x_2, \ldots, x_n$ be (not necessarily distinct) elements. The **ordered n-tuple, $(x_1, x_2, \ldots, x_n)$,** consists of $x_1, x_2, \ldots, x_n$ together with the ordering: first x_1, then x_2, and so forth up to x_n. An ordered 2-tuple is called an **ordered pair,** and an ordered 3-tuple is called an **ordered triple.**

Two ordered n-tuples $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ are **equal** if, and only if, $x_1 = y_1, x_2 = y_2, \ldots, x_n = y_n$.

Symbolically:

$$(x_1, x_2, \ldots, x_n) = (y_1, y_2, \ldots, y_n) \quad \Leftrightarrow \quad x_1 = y_1, x_2 = y_2, \ldots, x_n = y_n.$$

In particular,

$$(a, b) = (c, d) \quad \Leftrightarrow \quad a = c \text{ and } b = d.$$

Example 13 – *Ordered n-tuples*

a. Is $(1, 2, 3, 4) = (1, 2, 4, 3)$?

b. Is $\left(3, (-2)^2, \frac{1}{2}\right) = \left(\sqrt{9}, 4, \frac{3}{6}\right)$?

Solution:

a. No. By definition of equality of ordered 4-tuples,

$$(1,\ 2,\ 3,\ 4) = (1,\ 2,\ 4,\ 3) \Leftrightarrow 1 = 1,\ 2 = 2,\ 3 = 4,\ \text{and } 4 = 3$$

But $3 \neq 4$, and so the ordered 4-tuples are not equal.

Example 13 – *Solution*

cont'd

b. Yes. By definition of equality of ordered triples,

$$\left(3, (-2)^2, \tfrac{1}{2}\right) = \left(\sqrt{9}, 4, \tfrac{3}{6}\right) \quad \Leftrightarrow \quad 3 = \sqrt{9} \text{ and } (-2)^2 = 4 \text{ and } \tfrac{1}{2} = \tfrac{3}{6}.$$

Because these equations are all true, the two ordered triples are equal.

Cartesian Products

- **Definition**

Given sets $A_1, A_2, \ldots, A_n$, the **Cartesian product** of $A_1, A_2, \ldots, A_n$ denoted $\boldsymbol{A_1 \times A_2 \times \ldots \times A_n}$, is the set of all ordered n-tuples $(a_1, a_2, \ldots, a_n)$ where $a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n$.

Symbolically:

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n\}.$$

In particular,

$$A_1 \times A_2 = \{(a_1, a_2) \mid a_1 \in A_1 \text{ and } a_2 \in A_2\}$$

is the Cartesian product of A_1 and A_2.

Example 14 – *Cartesian Products*

Let $A_1 = \{x, y\}$, $A_2 = \{1, 2, 3\}$, and $A_3 = \{a, b\}$.

a. Find $A_1 \times A_2$.

b. Find $(A_1 \times A_2) \times A_3$.

c. Find $A_1 \times A_2 \times A_3$.

Solution:

a. $A_1 \times A_2 = \{(x, 1), (x, 2), (x, 3), (y, 1), (y, 2), (y, 3)\}$

b. The Cartesian product of A_1 and A_2 is a set, so it may be used as one of the sets making up another Cartesian product. This is the case for $(A_1 \times A_2) \times A_3$.

Example 14 – *Solution*

cont'd

$$(A_1 \times A_2) \times A_3 = \{(u, v) \mid u \in A_1 \times A_2 \text{ and } v \in A_3\}$$ by definition of Cartesian product

$$= \{((x, 1), a), ((x, 2), a), ((x, 3), a), ((y, 1), a),$$

$$((y, 2), a), ((y, 3), a), ((x, 1), b), ((x, 2), b), ((x, 3), b),$$

$$((y, 1), b), ((y, 2), b), ((y, 3), b)\}$$

c. The Cartesian product $A_1 \times A_2 \times A_3$ is superficially similar to, but is not quite the same mathematical object as, $(A_1 \times A_2) \times A_3$. $(A_1 \times A_2) \times A_3$ is a set of ordered pairs of which one element is itself an ordered pair, whereas $A_1 \times A_2 \times A_3$ is a set of ordered triples.

Example 14 – *Solution*

cont'd

By definition of Cartesian product,

$$A_1 \times A_2 \times A_3 = \{(u, v, w) \mid u \in A_1, v \in A_2, \text{and } w \in A_3\}$$

$$= \{(x, 1, a), (x, 2, a), (x, 3, a), (y, 1, a), (y, 2, a),$$

$$(y, 3, a), (x, 1, b), (x, 2, b), (x, 3, b), (y, 1, b),$$

$$(y, 2, b), (y, 3, b)\}.$$

An Algorithm to Check Whether One Set Is a Subset of Another (Optional)

An Algorithm to Check Whether One Set Is a Subset of Another (Optional)

Order the elements of both sets and successively compare each element of the first set with each element of the second set.

If some element of the first set is not found to equal any element of the second, then the first set is not a subset of the second.

But if each element of the first set is found to equal an element of the second set, then the first set is a subset of the second. The following algorithm formalizes this reasoning.

Algorithm 6.1.1 Testing Whether $A \subseteq B$:

[Input sets A and B are represented as one-dimensional arrays $a[1], a[2], \ldots, a[m]$ *and* $b[1], b[2], \ldots, b[n]$*, respectively. Starting with* $a[1]$ *and for each successive* $a[i]$ *in A, a check is made to see whether* $a[i]$ *is in B. To do this,* $a[i]$ *is compared to successive elements of B. If* $a[i]$ *is not equal to any element of B, then answer is given the value "*$A \not\subseteq B$*."*

If $a[i]$ *equals some element of B, the next successive element in A is checked to see whether it is in B. If every successive element of A is found to be in B, then answer never changes from its initial value "*$A \subseteq B$*."]*

An Algorithm to Check Whether One Set Is a Subset of Another (Optional)

Input:

m [*a positive integer*], *a*[1], *a*[2], . . . , *a*[*m*] [*a one-dimensional array representing the set A*], *n [a positive integer*], *b*[1], *b*[2], . . . , *b*[*n*] [*a one-dimensional array representing the set B*]

Algorithm Body:

i := 1, *answer* := "$A \subseteq B$"

  while (*i* ≤ *m and answer* = "$A \subseteq B$")

    j := 1, *found* := "*no*"

    while (***j*** **≤** ***n and found*** **= "*no*")**

      if *a*[*i*] = *b*[*j*] *then found* := "*yes*"

An Algorithm to Check Whether One Set Is a Subset of Another (Optional)

$j := j + 1$

end while

[*If found has not been given the value "yes" when execution reaches this point, then* $a[i] \notin B$.]

if *found* = *"no" then answer* := "$A \not\subseteq B$"

$i := i + 1$

end while

Output: *answer* [*a string*]

Example 15 – *Tracing Algorithm 6.1.1*

Trace the action of Algorithm 6.1.1 on the variables i, j, found, and answer for $m = 3$, $n = 4$, and sets A and B represented as the arrays $a[1] = u$, $a[2] = v$, $a[3] = w$, $b[1] = w$, $b[2] = x$, $b[3] = y$, and $b[4] = u$.

Solution:

i	1					2					3
j	1	2	3	4	5	1	2	3	4	5	
found	no			yes		no					
answer	$A \subseteq B$									$A \not\subseteq B$	