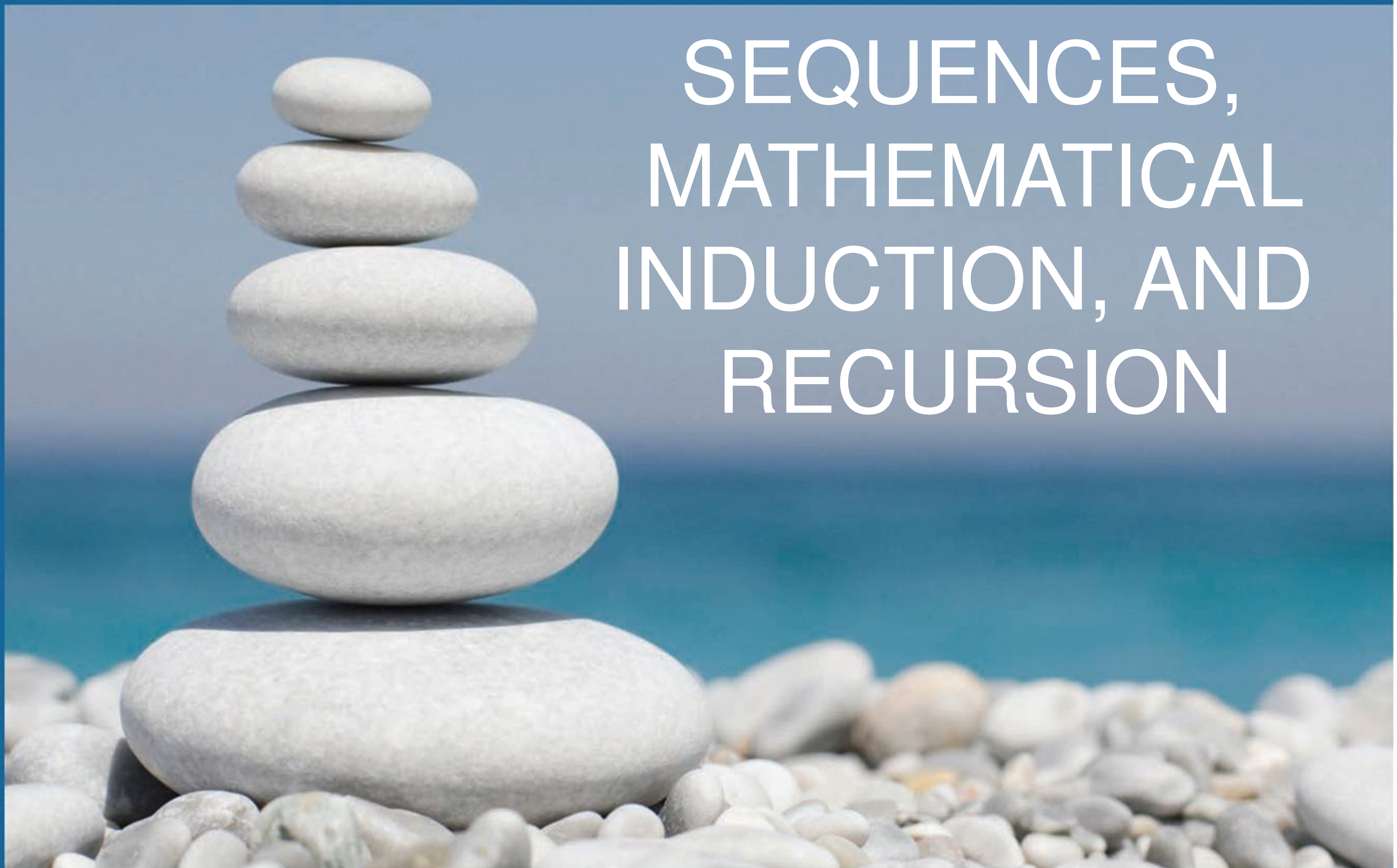
CHAPTER 5



# SEQUENCES, MATHEMATICAL INDUCTION, AND RECURSION

## SECTION 5.7

# Solving Recurrence Relations by Iteration

# Solving Recurrence Relations by Iteration

Suppose you have a sequence that satisfies a certain recurrence relation and initial conditions.

It is often helpful to know an explicit formula for the sequence, especially if you need to compute terms with very large subscripts or if you need to examine general properties of the sequence.

Such an explicit formula is called a **solution** to the recurrence relation. In this section, we discuss methods for solving recurrence relations.

# The Method of Iteration

# The Method of Iteration

The most basic method for finding an explicit formula for a recursively defined sequence is **iteration.**

Iteration works as follows: Given a sequence $a_0, a_1, a_2, \ldots$ defined by a recurrence relation and initial conditions, you start from the initial conditions and calculate successive terms of the sequence until you see a pattern developing.

At that point you guess an explicit formula.

# Example 1 – *Finding an Explicit Formula*

Let $a_0$, $a_1$, $a_2$, . . . be the sequence defined recursively as follows: For all integers $k \geq 1$,

(1) $a_k = a_{k-1} + 2$ recurrence relation

(2) $a_0 = 1$ initial condition.

Use iteration to guess an explicit formula for the sequence.

Solution:

We know that to say

$$a_k = a_{k-1} + 2 \quad \text{for all integers } k \geq 1$$

means

$$a_{\square} = a_{\square-1} + 2$$

no matter what positive integer is placed into the box $\square$.

# Example 1 – *Solution*

cont'd

In particular,

$$a_1 = a_0 + 2,$$

$$a_2 = a_1 + 2,$$

$$a_3 = a_2 + 2,$$

and so forth.

Now use the initial condition to begin a process of successive substitutions into these equations, not just of numbers but of *numerical expressions*.

# Example 1 – *Solution*

cont'd

The reason for using numerical expressions rather than numbers is that in these problems you are seeking a numerical pattern that underlies a general formula.

The secret of success is to leave most of the arithmetic undone.

However, you do need to eliminate parentheses as you go from one step to the next. Otherwise, you will soon end up with a bewilderingly large nest of parentheses.

# Example 1 – *Solution*

cont'd

Also, it is nearly always helpful to use shorthand notations for regrouping additions, subtractions, and multiplications of numbers that repeat.

Thus, for instance, you would write

$$5 \cdot 2 \qquad \text{instead of } 2 + 2 + 2 + 2 + 2$$

and

$$2^5 \qquad \text{instead of } 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2.$$

Notice that you don't lose any information about the number patterns when you use these shorthand notations.

# Example 1 – *Solution*

cont'd

Here's how the process works for the given sequence:

$a_0 = 1$ — the initial condition

$a_1 = a_0 + 2 = 1 + 2$ — by substitution

$a_2 = a_1 + 2 = (1 + 2) + 2 = 1 + 2 + 2$ — eliminate parentheses

$a_3 = a_2 + 2 = (1 + 2 + 2) + 2 = 1 + 2 + 2 + 2$ — eliminate parentheses again; write $3 \cdot 2$ instead of $2 + 2 + 2$?

$a_4 = a_3 + 2 = (1 + 2 + 2 + 2) + 2 = 1 + 2 + 2 + 2 + 2$ — eliminate parentheses again; definitely write $4 \cdot 2$ instead of $2 + 2 + 2 + 2$—the length of the string of 2's is getting out of hand.

# Example 1 – *Solution*

cont'd

Since it appears helpful to use the shorthand $k \cdot 2$ in place of $2 + 2 + \cdot \cdot \cdot + 2$ ($k$ times), we do so, starting again from $a_0$.

$$a_0 = 1 \qquad = 1 + 0 \cdot 2 \qquad \text{the initial condition}$$

$$a_1 = a_0 + 2 = 1 + 2 \qquad = 1 + 1 \cdot 2 \qquad \text{by substitution}$$

$$a_2 = a_1 + 2 = (1 + 2) + 2 \qquad = 1 + 2 \cdot 2$$

$$a_3 = a_2 + 2 = (1 + 2 \cdot 2) + 2 = 1 + 3 \cdot 2$$

# Example 1 – *Solution*

cont'd

$$a_3 = a_2 + 2 = \overbrace{(1 + 2 \cdot 2)} + 2 = \underbrace{1 + 3 \cdot 2}$$

$$a_4 = a_3 + 2 = \overbrace{(1 + 3 \cdot 2)} + 2 = \underbrace{1 + 4 \cdot 2}$$

$$a_5 = a_4 + 2 = \overbrace{(1 + 4 \cdot 2)} + 2 = 1 + 5 \cdot 2$$

$$\vdots$$

At this point it certainly seems likely that the general pattern is $1 + n \cdot 2$; check whether the next calculation supports this.

It does! So go ahead and write an answer. It's only a guess, after all.

*Guess*: $a_n = 1 + n \cdot 2 = 1 + 2n$

The answer obtained for this problem is just a guess. To be sure of the correctness of this guess, you will need to check it by mathematical induction.

# The Method of Iteration

A sequence like the one in Example 1, in which each term equals the previous term plus a fixed constant, is called an *arithmetic sequence*.

**• Definition**

A sequence $a_0,\ a_1,\ a_2, \ldots$ is called an **arithmetic sequence** if, and only if, there is a constant $d$ such that

$$a_k = a_{k-1} + d \quad \text{for all integers } k \geq 1.$$

It follows that,

$$a_n = a_0 + dn \quad \text{for all integers } n \geq 0.$$

# Example 2 – *An Arithmetic Sequence*

Under the force of gravity, an object falling in a vacuum falls about 9.8 meters per second (m/sec) faster each second than it fell the second before.

Thus, neglecting air resistance, a skydiver's speed upon leaving an airplane is approximately 9.8m/sec one second after departure, 9.8 + 9.8 = 19.6m/sec two seconds after departure, and so forth.

If air resistance is neglected, how fast would the skydiver be falling 60 seconds after leaving the airplane?

# Example 2 – *Solution*

Let $s_n$ be the skydiver's speed in m/sec $n$ seconds after exiting the airplane if there were no air resistance.

Thus $s_0$ is the initial speed, and since the diver would travel 9.8m/sec faster each second than the second before,

$$s_k = s_{k-1} + 9.8 \text{ m/sec} \quad \text{for all integers } k \geq 1.$$

It follows that $s_0$, $s_1$, $s_2$, . . . is an arithmetic sequence with a fixed constant of 9.8, and thus

$$s_n = s_0 + (9.8)n \quad \text{for each integer } n \geq 0.$$

# Example 2 – *Solution*

cont'd

Hence sixty seconds after exiting and neglecting air resistance, the skydiver would travel at a speed of

$$s_{60} = 0 + (9.8)(60) = 588 \text{ m/sec.}$$

Note that 588 m/sec is over half a kilometer per second or over a third of a mile per second, which is very fast for a human being to travel.

Happily for the skydiver, taking air resistance into account cuts the speed considerably.

# The Method of Iteration

Let $r$ be a fixed nonzero constant, and suppose a sequence $a_0, a_1, a_2, \ldots$ is defined recursively as follows:

$$a_k = ra_{k-1} \quad \text{for all integers } k \geq 1,$$

$$a_0 = a.$$

Use iteration to guess an explicit formula for this sequence.

**• Definition**

A sequence $a_0, a_1, a_2, \ldots$ is called a **geometric sequence** if, and only if, there is a constant $r$ such that

$$a_k = ra_{k-1} \quad \text{for all integers } k \geq 1.$$

It follows that,

$$a_n = a_0 r^n \quad \text{for all integers } n \geq 0.$$

# The Method of Iteration

An important property of a geometric sequence with constant multiplier greater than 1 is that its terms increase very rapidly in size as the subscripts get larger and larger.

For instance, the first ten terms of a geometric sequence with a constant multiplier of 10 are

$$1,\ 10,\ 10^2,\ 10^3,\ 10^4,\ 10^5,\ 10^6,\ 10^7,\ 10^8,\ 10^9.$$

Thus, by its tenth term, the sequence already has the value $10^9$ = 1,000,000,000 = 1 billion.

# The Method of Iteration

The following box indicates some quantities that are approximately equal to certain powers of 10.

$10^7 \cong$ number of seconds in a year

$10^9 \cong$ number of bytes of memory in a personal computer

$10^{11} \cong$ number of neurons in a human brain

$10^{17} \cong$ age of the universe in seconds (according to one theory)

$10^{31} \cong$ number of seconds to process all possible positions of a checkers game if moves are processed at a rate of 1 per billionth of a second

$10^{81} \cong$ number of atoms in the universe

$10^{111} \cong$ number of seconds to process all possible positions of a chess game if moves are processed at a rate of 1 per billionth of a second

# Using Formulas to Simplify Solutions Obtained by Iteration

## Using Formulas to Simplify Solutions Obtained by Iteration

Explicit formulas obtained by iteration can often be simplified by using formulas such as those developed earlier.

For instance, according to the formula for the sum of a geometric sequence with initial term 1 (Theorem 5.2.3), for each real number *r* except *r* = 1,

$$1 + r + r^2 + \cdots + r^n = \frac{r^{n+1} - 1}{r - 1} \quad \text{for all integers } n \geq 0.$$

**Theorem 5.2.3 Sum of a Geometric Sequence**

For any real number $r$ except 1, and any integer $n \geq 0$,

$$\sum_{i=0}^{n} r^i = \frac{r^{n+1} - 1}{r - 1}.$$

# Using Formulas to Simplify Solutions Obtained by Iteration

And according to the formula for the sum of the first *n* integers (Theorem 5.2.2),

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2} \quad \text{for all integers } n \geq 1.$$

**Theorem 5.2.2 Sum of the First *n* Integers**

For all integers $n \geq 1$,

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

## Example 5 – *An Explicit Formula for the Tower of Hanoi Sequence*

The Tower of Hanoi sequence $m_1$, $m_2$, $m_3$, . . . satisfies the recurrence relation

$$m_k = 2m_{k-1} + 1 \quad \text{for all integers } k \geq 2$$

and has the initial condition

$$m_1 = 1.$$

Use iteration to guess an explicit formula for this sequence, to simplify the answer.

# Example 5 – *Solution*

By iteration

$$m_1 = 1$$

$$m_{②} = 2m_1 + 1 = 2 \cdot 1 + 1 \qquad = 2^{①} + 1,$$

$$m_{③} = 2m_2 + 1 = 2\,(2 + 1) + 1 \qquad = 2^{②} + 2 + 1,$$

$$m_{④} = 2m_3 + 1 = 2\,(2^2 + 2 + 1) + 1 \qquad = 2^{③} + 2^2 + 2 + 1,$$

$$m_{⑤} = 2m_4 + 1 = 2\,(2^3 + 2^2 + 2 + 1) + 1 = 2^{④} + 2^3 + 2^2 + 2 + 1.$$

# Example 5 – *Solution*

cont'd

These calculations show that each term up to $m_5$ is a sum of successive powers of 2, starting with $2^0 = 1$ and going up to $2^k$, where $k$ is 1 less than the subscript of the term.

The pattern would seem to continue to higher terms because each term is obtained from the preceding one by multiplying by 2 and adding 1; multiplying by 2 raises the exponent of each component of the sum by 1, and adding 1 adds back the 1 that was lost when the previous 1 was multiplied by 2.

For instance, for $n = 6$,

$$m_6 = 2m_5 + 1 = 2(2^4 + 2^3 + 2^2 + 2 + 1) + 1 = 2^5 + 2^4 + 2^3 + 2^2 + 2 + 1.$$

# Example 5 – *Solution*

cont'd

Thus it seems that, in general,

$$m_n = 2^{n-1} + 2^{n-2} + \cdots + 2^2 + 2 + 1.$$

By the formula for the sum of a geometric sequence (Theorem 5.2.3),

**Theorem 5.2.3 Sum of a Geometric Sequence**

For any real number $r$ except 1, and any integer $n \geq 0$,

$$\sum_{i=0}^{n} r^i = \frac{r^{n+1} - 1}{r - 1}.$$

$$2^{n-1} + 2^{n-2} + \cdots + 2^2 + 2 + 1 = \frac{2^n - 1}{2 - 1} = 2^n - 1.$$

# Example 5 – *Solution*

cont'd

Hence the explicit formula seems to be

$$m_n = 2^n - 1 \quad \text{for all integers } n \geq 1.$$

# Checking the Correctness of a Formula by Mathematical Induction

## Checking the Correctness of a Formula by Mathematical Induction

It is all too easy to make a mistake and come up with the wrong formula.

That is why it is important to confirm your calculations by checking the correctness of your formula.

The most common way to do this is to use mathematical induction.

## Example 7 – *Using Mathematical Induction to Verify the Correctness of a Solution to a Recurrence Relation*

In 1883 a French mathematician, Édouard Lucas, invented a puzzle that he called The Tower of Hanoi (La Tour D’Hanoï).

The puzzle consisted of eight disks of wood with holes in their centers, which were piled in order of decreasing size on one pole in a row of three.

Those who played the game were supposed to move all the disks one by one from one pole to another, never placing a larger disk on top of a smaller one.

## Example 7 – *Using Mathematical Induction to Verify the Correctness of a Solution to a Recurrence Relation*

cont'd

The puzzle offered a prize of ten thousand francs (about $34,000 US today) to anyone who could move a tower of 64 disks by hand while following the rules of the game.

(See Figure 5.6.2) Assuming that you transferred the disks as efficiently as possible, how many moves would be required to win the prize?

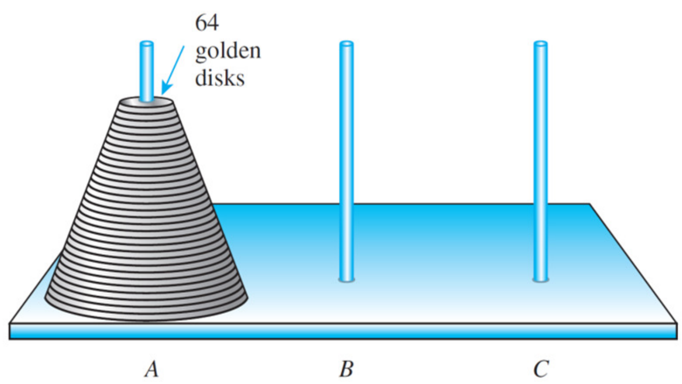



**Figure 5.6.2**

## Example 7 – *Using Mathematical Induction to Verify the Correctness of a Solution to a Recurrence Relation*

cont'd

The solution to this is as follows:

Let *m* be the minimum number of moves needed to transfer a tower of *k* disks from one pole to another. Then,

> If $m_1, m_2, m_3, \ldots$ is the sequence defined by
>
> $$m_k = 2m_{k-1} + 1 \quad \text{for all integers } k \geq 2, \text{ and}$$
> $$m_1 = 1,$$
>
> then $m_n = 2^n - 1$ for all integers $n \geq 1$.

Use mathematical induction to show that this formula is correct.

# Example 7 – *Solution*

What does it mean to show the correctness of a formula for a recursively defined sequence? Given a sequence of numbers that satisfies a certain recurrence relation and initial condition, your job is to show that each term of the sequence satisfies the proposed explicit formula.

In this case, you need to prove the following statement:

If $m_1, m_2, m_3, \ldots$ is the sequence defined by

$$m_k = 2m_{k-1} + 1 \quad \text{for all integers } k \geq 2, \text{ and}$$
$$m_1 = 1,$$

then $m_n = 2^n - 1$ for all integers $n \geq 1$.

# Example 7 – *Solution*

cont'd

**Proof of Correctness:**

Let $m_1, m_2, m_3, \ldots$ be the sequence defined by specifying that $m_1 = 1$ and $m_k = 2m_{k+1} + 1$ for all integers $k \geq 2$, and let the property $P(n)$ be the equation

$$m_n = 2^n - 1 \qquad \leftarrow P(n)$$

We will use mathematical induction to prove that for all integers $n \geq 1$, $P(n)$ is true.

**Show that *P*(1) is true:**

To establish $P(1)$, we must show that

$$m_1 = 2^1 - 1. \qquad \leftarrow P(1)$$

# Example 7 – *Solution*

cont'd

But the left-hand side of $P(1)$ is

$$m_1 = 1 \qquad \text{by definition of } m_1, m_2, m_3, \ldots,$$

and the right-hand side of $P(1)$ is

$$2^1 - 1 = 2 - 1 = 1.$$

Thus the two sides of $P(1)$ equal the same quantity, and hence $P(1)$ is true.

# Example 7 – *Solution*

cont'd

**Show that for all integers $k \geq 1$, if $P(k)$ is true then $P(k + 1)$ is also true:**

*[Suppose that* $P(k)$ *is true for a particular but arbitrarily chosen integer* $k \geq 1$. *That is:]*

Suppose that $k$ is any integer with $k \geq 1$ such that

$$m_k = 2^k - 1. \qquad \leftarrow P(k) \text{ inductive hypothesis}$$

*[We must show that* $P(k + 1)$ *is true. That is:]*

We must show that

$$m_{k+1} = 2^{k+1} - 1. \qquad \leftarrow P(k+1)$$

# Example 7 – *Solution*

cont'd

But the left-hand side of $P(k + 1)$ is

$$
\begin{aligned}
m_{k+1} &= 2m_{(k+1)-1} + 1 && \text{by definition of } m_1,\ m_2,\ m_3, \ldots \\
&= 2m_k + 1 \\
&= 2(2^k - 1) + 1 && \text{by substitution from the inductive hypothesis} \\
&= 2^{k+1} - 2 + 1 && \text{by the distributive law and the fact that } 2 \cdot 2^k = 2^{k-1} \\
&= 2^{k+1} - 1 && \text{by basic algebra}
\end{aligned}
$$

which equals the right-hand side of $P(k + 1)$. *[Since the basis and inductive steps have been proved, it follows by mathematical induction that the given formula holds for all integers $n \geq 1$.]*

# Discovering That an Explicit Formula Is Incorrect

# Discovering That an Explicit Formula Is Incorrect

The next example shows how the process of trying to verify a formula by mathematical induction may reveal a mistake.

## Example 8 – *Using Verification by Mathematical Induction to Find a Mistake*

Let $c_0, c_1, c_2, \ldots$ be the sequence defined as follows:

$$c_k = 2c_{k-1} + k \qquad \text{for all integers } k \geq 1,$$

$$c_0 = 1.$$

Suppose your calculations suggest that $c_0, c_1, c_2, \ldots$ satisfies the following explicit formula:

$$c_n = 2^n + n \qquad \text{for all integers } n \geq 0.$$

Is this formula correct?

# Example 8 – *Solution*

Start to prove the statement by mathematical induction and see what develops.

The proposed formula passes the basis step of the inductive proof with no trouble, for on the one hand, $c_0 = 1$ by definition and on the other hand, $2^0 + 0 = 1 + 0 = 1$ also.

In the inductive step, you suppose

$$c_k = 2^k + k \qquad \text{for some integer } k \geq 0$$

This is the inductive hypothesis.

and then you must show that

$$c_{k+1} = 2^{k+1} + (k+1).$$

# Example 8 – *Solution*

cont'd

To do this, you start with $c_{k+1}$, substitute from the recurrence relation, and then use the inductive hypothesis as follows:

$$c_{k+1} = 2c_k + (k + 1) \qquad \text{by the recurrence relation}$$

$$= 2(2^k + k) + (k + 1) \qquad \text{by substitution from the inductive hypothesis}$$

$$= 2^{(k+1)} + 3k + 1 \qquad \text{by basic algebra}$$

To finish the verification, therefore, you need to show that

$$2^{k+1} + 3k + 1 = 2^{k+1} + (k + 1).$$

# Example 8 – *Solution*

cont'd

Now this equation is equivalent to

$$2k = 0 \qquad \text{by subtracting } 2^{k+1} + k + 1 \text{ from both sides.}$$

which is equivalent to

$$k = 0 \qquad \text{by dividing both sides by 2.}$$

But this is false since *k* may be *any* nonnegative integer.

Observe that when $k = 0$, then $k + 1 = 1$, and

$$c_1 = 2 \cdot 1 + 1 = 3 \quad \text{and} \quad 2^1 + 1 = 3.$$

# Example 8 – *Solution*

cont'd

Thus the formula gives the correct value for $c_1$. However, when $k = 1$, then $k + 1 = 2$, and

$$c_2 = 2 \cdot 3 + 2 = 8 \quad \text{whereas} \quad 2^2 + 2 = 4 + 2 = 6.$$

So the formula does not give the correct value for $c_2$. Hence the sequence $c_0, c_1, c_2, \ldots$ does not satisfy the proposed formula.

# Discovering That an Explicit Formula Is Incorrect

Once you have found a proposed formula to be false, you should look back at your calculations to see where you made a mistake, correct it, and try again.