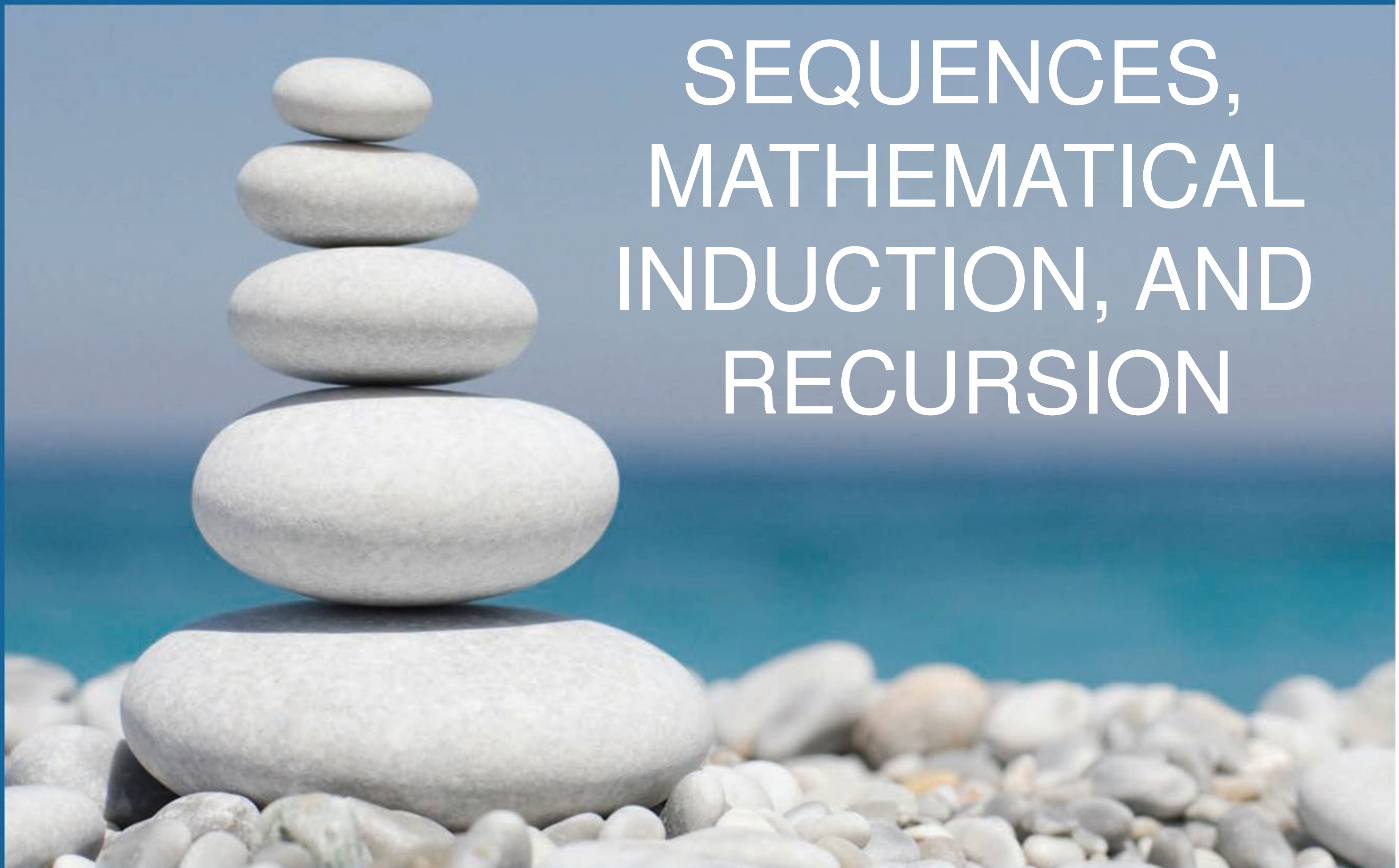
CHAPTER 5



SEQUENCES, MATHEMATICAL INDUCTION, AND RECURSION

SECTION 5.2

Mathematical Induction I

Mathematical Induction I

Mathematical induction is one of the more recently developed techniques of proof in the history of mathematics.

It is used to check conjectures about the outcomes of processes that occur repeatedly and according to definite patterns.

In general, mathematical induction is a method for proving that a property defined for integers n is true for all values of n that are greater than or equal to some initial integer.

Mathematical Induction I

Principle of Mathematical Induction

Let $P(n)$ be a property that is defined for integers n, and let a be a fixed integer. Suppose the following two statements are true:

1. $P(a)$ is true.
2. For all integers $k \geq a$, if $P(k)$ is true then $P(k+1)$ is true.

Then the statement

$$\text{for all integers } n \geq a, P(n)$$

is true.

The validity of proof by mathematical induction is generally taken as an axiom. That is why it is referred to as the *principle* of mathematical induction rather than as a theorem.

Mathematical Induction I

Proving a statement by mathematical induction is a two-step process. The first step is called the *basis step*, and the second step is called the *inductive step*.

Method of Proof by Mathematical Induction

Consider a statement of the form, "For all integers $n \geq a$, a property $P(n)$ is true." To prove such a statement, perform the following two steps:

Step 1 (basis step): Show that $P(a)$ is true.

Step 2 (inductive step): Show that for all integers $k \geq a$, if $P(k)$ is true then $P(k+1)$ is true. To perform this step,

suppose that $P(k)$ is true, where k is any particular but arbitrarily chosen integer with $k \geq a$.

[This supposition is called the **inductive hypothesis.***]*

Then

show that $P(k+1)$ is true.

Mathematical Induction I

The following example shows how to use mathematical induction to prove a formula for the sum of the first *n* integers.

Example 1 – *Sum of the First n Integers*

Use mathematical induction to prove that

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2} \quad \text{for all integers } n \geq 1.$$

Solution:

To construct a proof by induction, you must first identify the property *P*(*n*). In this case, *P*(*n*) is the equation

$$\boxed{1 + 2 + \cdots + n = \frac{n(n+1)}{2}.} \qquad \leftarrow \text{the property } (P(n))$$

[To see that P(n) is a sentence, note that its subject is "the sum of the integers from 1 to n" and its verb is "equals."]

Example 1 – *Solution*

cont'd

In the basis step of the proof, you must show that the property is true for $n = 1$, or, in other words that $P(1)$ is true.

Now $P(1)$ is obtained by substituting 1 in place of n in $P(n)$. The left-hand side of $P(1)$ is the sum of all the successive integers starting at 1 and ending at 1. This is just 1. Thus $P(1)$ is

$$\boxed{1 = \frac{1(1+1)}{2}.} \qquad \leftarrow \text{basis } (P(1))$$

Example 1 – *Solution*

cont'd

Of course, this equation is true because the right-hand side is

$$\frac{1(1+1)}{2} = \frac{1 \cdot 2}{2} = 1,$$

which equals the left-hand side.

In the inductive step, you assume that *P*(*k*) is true, for a particular but arbitrarily chosen integer *k* with $k \geq 1$. *[This assumption is the inductive hypothesis.]*

Example 1 – *Solution*

cont'd

You must then show that $P(k + 1)$ is true. What are $P(k)$ and $P(k + 1)$? $P(k)$ is obtained by substituting k for every n in $P(n)$.

Thus $P(k)$ is

$$\boxed{1 + 2 + \cdots + k = \frac{k(k+1)}{2}.} \qquad \leftarrow \text{inductive hypothesis } (P(k))$$

Similarly, $P(k + 1)$ is obtained by substituting the quantity $(k + 1)$ for every n that appears in $P(n)$.

Example 1 – *Solution*

cont'd

Thus $P(k + 1)$ is

$$1 + 2 + \cdots + (k + 1) = \frac{(k + 1)((k + 1) + 1)}{2},$$

or, equivalently,

$$\boxed{1 + 2 + \cdots + (k + 1) = \frac{(k + 1)(k + 2)}{2}.}$$

$\leftarrow$ to show $(P(k + 1))$

Example 1 – *Solution*

cont'd

Now the inductive hypothesis is the supposition that $P(k)$ is true. How can this supposition be used to show that $P(k + 1)$ is true? $P(k + 1)$ is an equation, and the truth of an equation can be shown in a variety of ways.

One of the most straightforward is to use the inductive hypothesis along with algebra and other known facts to transform separately the left-hand and right-hand sides until you see that they are the same.

Example 1 – *Solution*

cont'd

In this case, the left-hand side of $P(k + 1)$ is

$$1 + 2 + \cdots + (k + 1),$$

which equals

$$(1 + 2 + \cdots + k) + (k + 1)$$

The next-to-last term is k because the terms are successive integers and the last term is $k + 1$.

But by substitution from the inductive hypothesis,

$$(1 + 2 + \cdots + k) + (k + 1)$$

$$= \frac{k(k+1)}{2} + (k + 1)$$

since the inductive hypothesis says that $1 + 2 + \cdots + k = \frac{k(k+1)}{2}$

Example 1 – *Solution*

cont'd

$$= \frac{k(k+1)}{2} + \frac{2(k+1)}{2}$$

by multiplying the numerator and denominator of the second term by 2 to obtain a common denominator

$$= \frac{k^2+k}{2} + \frac{2k+2}{2}$$

by multiplying out the two numerators

$$= \frac{k^2+3k+2}{2}$$

by adding fractions with the same denominator and combining like terms.

Example 1 – *Solution*

cont'd

So the left-hand side of $P(k + 1)$ is $\dfrac{k^2 + 3k + 2}{2}$.

Now the right-hand side of $P(k + 1)$ is $\dfrac{(k+1)(k+2)}{2} = \dfrac{k^2 + 3k + 2}{2}$ by multiplying out the numerator.

Thus the two sides of $P(k + 1)$ are equal to each other, and so the equation $P(k + 1)$ is true.

This discussion is summarized as follows:

Theorem 5.2.2 Sum of the First *n* Integers

For all integers $n \geq 1$,

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

Example 1 – *Solution*

cont'd

Proof (by mathematical induction):

Let the property *P*(*n*) be the equation

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}. \qquad \leftarrow P(n)$$

Show that *P*(1) is true:

To establish *P*(1), we must show that

$$1 = \frac{1(1+1)}{2} \qquad \leftarrow \quad P(1)$$

Example 1 – *Solution*

cont'd

But the left-hand side of this equation is 1 and the right-hand side is

$$\frac{1(1+1)}{2} = \frac{2}{2} = 1$$

also. Hence *P*(1) is true.

Show that for all integers *k* ≥ 1, if *P*(*k*) is true then *P*(*k* + 1) is also true:

[Suppose that P(k) is true for a particular but arbitrarily chosen integer k ≥ 1. That is:] Suppose that *k* is any integer with *k* ≥ 1 such that

$$1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2} \qquad \leftarrow P(k) \text{ inductive hypothesis}$$

Example 1 – *Solution*

cont'd

[We must show that $P(k + 1)$ *is true. That is:]* We must show that

$$1 + 2 + 3 + \cdots + (k + 1) = \frac{(k + 1)[(k + 1) + 1]}{2},$$

or, equivalently, that

$$1 + 2 + 3 + \cdots + (k + 1) = \frac{(k + 1)(k + 2)}{2}. \qquad \leftarrow P(k + 1)$$

[We will show that the left-hand side and the right-hand side of $P(k + 1)$ *are equal to the same quantity and thus are equal to each other.]*

Example 1 – *Solution*

cont'd

The left-hand side of $P(k + 1)$ is

$$1 + 2 + 3 + \cdots + (k + 1)$$

$$= 1 + 2 + 3 + \cdots + k + (k + 1)$$ by making the next-to-last term explicit

$$= \frac{k(k+1)}{2} + (k + 1)$$ by substitution from the inductive hypothesis

$$= \frac{k(k+1)}{2} + \frac{2(k+1)}{2}$$

$$= \frac{k^2 + k}{2} + \frac{2k + 2}{2}$$

Example 1 – *Solution*

cont'd

$$= \frac{k^2 + 3k + 2}{2} \qquad \text{by algebra.}$$

And the right-hand side of $P(k + 1)$ is

$$\frac{(k+1)(k+2)}{2} = \frac{k^2 + 3k + 2}{2}.$$

Example 1 – *Solution*

cont'd

Thus the two sides of $P(k + 1)$ are equal to the same quantity and so they are equal to each other. Therefore the equation $P(k + 1)$ is true *[as was to be shown]*.

[Since we have proved both the basis step and the inductive step, we conclude that the theorem is true.]

Mathematical Induction I

• Definition Closed Form

If a sum with a variable number of terms is shown to be equal to a formula that does not contain either an ellipsis or a summation symbol, we say that it is written **in closed form**.

For example, writing $1 + 2 + 3 + \cdots + n = \dfrac{n(n+1)}{2}$ expresses the sum $1 + 2 + 3 + \cdots + n$ in closed form.

Example 2 – *Applying the Formula for the Sum of the First n Integers*

a. Evaluate $2 + 4 + 6 + \cdots + 500$.

b. Evaluate $5 + 6 + 7 + 8 + \cdots + 50$.

c. For an integer $h \geq 2$, write $1 + 2 + 3 + \cdots + (h - 1)$ in closed form.

Example 2 – *Solution*

a. $2 + 4 + 6 + \cdots + 500 = 2 \cdot (1 + 2 + 3 + \cdots + 250)$

$$= 2 \cdot \left(\frac{250 \cdot 251}{2}\right)$$

by applying the formula for the sum of the first n integers with $n = 250$

$$= 62{,}750.$$

b. $5 + 6 + 7 + 8 + \cdots + 50 = (1 + 2 + 3 + \cdots + 50) - (1 + 2 + 3 + 4)$

$$= \frac{50 \cdot 51}{2} - 10$$

by applying the formula for the sum of the first n integers with $n = 50$

$$= 1{,}265$$

Example 2 – *Solution*

cont'd

c. $$1 + 2 + 3 + \cdots + (h - 1) = \frac{(h-1)\cdot[(h-1)+1]}{2}$$ by applying the formula for the sum of the first n integers with $n = h - 1$

$$= \frac{(h-1)\cdot h}{2}$$ since $(h - 1) + 1 = h$.

Mathematical Induction I

In a **geometric sequence,** each term is obtained from the preceding one by multiplying by a constant factor.

If the first term is 1 and the constant factor is r, then the sequence is 1, r, r^2, r^3, . . . , r^n, ...

The sum of the first n terms of this sequence is given by the formula

$$\sum_{i=0}^{n} r^i = \frac{r^{n+1} - 1}{r - 1}$$

for all integers $n \geq 0$ and real numbers r not equal to 1.

Mathematical Induction I

The expanded form of the formula is

$$r^0 + r^1 + r^2 + \cdots + r^n = \frac{r^{n+1} - 1}{r - 1},$$

and because $r^0 = 1$ and $r^1 = r$, the formula for $n \geq 1$ can be rewritten as

$$1 + r + r^2 + \cdots + r^n = \frac{r^{n+1} - 1}{r - 1}.$$

Example 3 – *Sum of a Geometric Sequence*

Prove that $\sum_{i=0}^{n} r^i = \frac{r^{n+1} - 1}{r - 1}$, for all integers $n \geq 0$ and all real numbers r except 1.

Solution:

In this example the property $P(n)$ is again an equation, although in this case it contains a real variable r:

$$\boxed{\sum_{i=0}^{n} r^i = \frac{r^{n+1} - 1}{r - 1}.} \qquad \leftarrow \text{the property } (P(n))$$

Example 3 – *Solution*

cont'd

Because r can be any real number other than 1, the proof begins by supposing that r is a particular but arbitrarily chosen real number not equal to 1.

Then the proof continues by mathematical induction on n, starting with $n = 0$.

In the basis step, you must show that $P(0)$ is true; that is, you show the property is true for $n = 0$.

Example 3 – *Solution*

cont'd

So you substitute 0 for each *n* in *P*(*n*)*:*

$$\boxed{\sum_{i=0}^{0} r^i = \frac{r^{0+1} - 1}{r - 1}.} \qquad \leftarrow \text{basis } (P(0))$$

In the inductive step, you suppose *k* is any integer with $k \geq 0$ for which *P*(*k*) is true; that is, you suppose the property is true for $n = k$.

Example 3 – *Solution*

cont'd

So you substitute k for each n in $P(n)$:

$$\boxed{\sum_{i=0}^{k} r^i = \frac{r^{k+1}-1}{r-1}.}$$

$\leftarrow$ inductive hypothesis ($P(k)$)

Then you show that $P(k + 1)$ is true; that is, you show the property is true for $n = k + 1$.

So you substitute $k + 1$ for each n in $P(n)$:

$$\sum_{i=0}^{k+1} r^i = \frac{r^{(k+1)+1}-1}{r-1},$$

Example 3 – *Solution*

cont'd

Or, equivalently,

$$\boxed{\sum_{i=0}^{k+1} r^i = \frac{r^{k+2}-1}{r-1}.} \qquad \leftarrow \text{ to show } (P(k+1))$$

In the inductive step for this proof we use another common technique for showing that an equation is true:

We start with the left-hand side and transform it step-by-step into the right-hand side using the inductive hypothesis together with algebra and other known facts.

Example 3 – *Solution*

cont'd

Theorem 5.2.3 Sum of a Geometric Sequence

For any real number r except 1, and any integer $n \geq 0$,

$$\sum_{i=0}^{n} r^i = \frac{r^{n+1} - 1}{r - 1}.$$

Proof (by mathematical induction):
Suppose *r* is a particular but arbitrarily chosen real number that is not equal to 1, and let the property *P*(*n*) be the equation

$$\sum_{i=0}^{n} r^i = \frac{r^{i+1} - 1}{r - 1} \qquad \leftarrow P(n)$$

We must show that *P*(*n*) is true for all integers $n \geq 0$. We do this by mathematical induction on *n*.

Example 3 – *Solution*

cont'd

Show that *P*(0) is true:

To establish *P*(0), we must show that

$$\sum_{i=0}^{0} r^i = \frac{r^{0+1}-1}{r-1} \qquad \leftarrow P(0)$$

The left-hand side of this equation is $r^0 = 1$ and the right-hand side is

$$\frac{r^{0+1}-1}{r-1} = \frac{r-1}{r-1} = 1$$

also because $r^1 = r$ and $r \neq 1$. Hence *P*(0) is true.

Example 3 – *Solution*

cont'd

Show that for all integers $k \geq 0$, if $P(k)$ is true then $P(k + 1)$ is also true:

[Suppose that P(k) is true for a particular but arbitrarily chosen integer k ≥ 0. That is:]

Let k be any integer with $k \geq 0$, and suppose that

$$\sum_{i=0}^{k} r^i = \frac{r^{k+1} - 1}{r - 1} \qquad \leftarrow P(k) \text{ inductive hypothesis}$$

[We must show that P(k + 1) is true. That is:] We must show that

$$\sum_{i=0}^{k+1} r^i = \frac{r^{(k+1)+1} - 1}{r - 1}.$$

Example 3 – *Solution*

cont'd

Or, equivalently, that

$$\sum_{i=0}^{k+1} r^i = \frac{r^{k+2} - 1}{r - 1}. \qquad \leftarrow P(k+1)$$

[We will show that the left-hand side of $P(k + 1)$ *equals the right-hand side.]* The left-hand side of $P(k + 1)$ is

$$\sum_{i=0}^{k+1} r^i = \sum_{i=0}^{k} r^i + r^{k+1}$$

by writing the $(k + 1)$st term separately from the first k terms

$$= \frac{r^{k+1} - 1}{r - 1} + r^{k+1}$$

by substitution from the inductive hypothesis

Example 3 – *Solution*

cont'd

$$= \frac{r^{k+1} - 1}{r - 1} + \frac{r^{k+1}(r - 1)}{r - 1}$$

by multiplying the numerator and denominator of the second term by $(r - 1)$ to obtain a common denominator

$$= \frac{(r^{k+1} - 1) + r^{k+1}(r - 1)}{r - 1}$$

by adding fractions

$$= \frac{r^{k+1} - 1 + r^{k+2} - r^{k+1}}{r - 1}$$

by multiplying out and using the fact that $r^{k+1} \cdot r = r^{k+1} \cdot r^1 = r^{k+2}$

$$= \frac{r^{k+2} - 1}{r - 1}$$

by canceling the r^{k+1}'s.

which is the right-hand side of $P(k + 1)$ *[as was to be shown.]*

[Since we have proved the basis step and the inductive step, we conclude that the theorem is true.]

Proving an Equality

Proving an Equality

The proofs of the basis and inductive steps in Examples 1 and 3 illustrate two different ways to show that an equation is true:

(1) transforming the left-hand side and the right-hand side independently until they are seen to be equal, and

(2) transforming one side of the equation until it is seen to be the same as the other side of the equation.

Sometimes people use a method that they believe proves equality but that is actually invalid.

Proving an Equality

For example, to prove the basis step for Theorem 5.2.3, they perform the following steps:

$$\sum_{i=0}^{0} r^i = \frac{r^{0+1} - 1}{r - 1}$$

$$r^0 = \frac{r^1 - 1}{r - 1}$$

$$1 = \frac{r - 1}{r - 1}$$

$$1 = 1$$

Proving an Equality

The problem with this method is that starting from a statement and deducing a true conclusion does not prove that the statement is true.

A true conclusion can also be deduced from a false statement. For instance, the steps below show how to deduce the true conclusion that 1 = 1 from the false statement that 1 = 0:

$$1 = 0 \qquad \leftarrow \text{false}$$

$$0 = 1$$

Proving an Equality

$$1 + 0 = 0 + 1$$

$$1 = 1 \qquad \leftarrow \text{true}$$

When using mathematical induction to prove formulas, be sure to use a method that avoids invalid reasoning, both for the basis step and for the inductive step.

Deducing Additional Formulas

Deducing Additional Formulas

The formula for the sum of a geometric sequence can be thought of as a family of different formulas in r, one for each real number r except 1.

Example 4 – *Applying the Formula for the Sum of a Geometric Sequence*

In each of (**a**) and (**b**) below, assume that *m* is an integer that is greater than or equal to 3. Write each of the sums in closed form.

a. $1 + 3 + 3^2 + \cdots + 3^{m-2}$

b. $3^2 + 3^3 + 3^4 + \cdots + 3^m$

Solution:

a. $$1 + 3 + 3^2 + \cdots + 3^{m-2} = \frac{3^{(m-2)+1} - 1}{3 - 1}$$ by applying the formula for the sum of a geometric sequence with $r = 3$ and $n = m - 2$

$$= \frac{3^{m-1} - 1}{2}.$$

Example 4 – *Solution*

cont'd

b. $3^2 + 3^3 + 3^4 + \cdots + 3^m = 3^2 \cdot (1 + 3 + 3^2 + \cdots + 3^{m-2})$ by factoring out 3^2

$$= 9 \cdot \left(\frac{3^{m-1} - 1}{2}\right)$$ by part (a).

Deducing Additional Formulas

As with the formula for the sum of the first *n* integers, there is a way to think of the formula for the sum of the terms of a geometric sequence that makes it seem simple and intuitive. Let

$$S_n = 1 + r + r^2 + \cdots + r^n.$$

Then

$$r S_n = r + r^2 + r^3 + \cdots + r^{n+1},$$

and so

$$r S_n - S_n = (r + r^2 + r^3 + \cdots + r^{n+1}) - (1 + r + r^2 + \cdots + r^n)$$

$$= r^{n+1} - 1.$$

5.2.1

Deducing Additional Formulas

But

$$rS_n - S_n = (r-1)S_n. \qquad 5.2.2$$

Equating the right-hand sides of equations (5.2.1) and (5.2.2) and dividing by $r - 1$ gives

$$S_n = \frac{r^{n+1} - 1}{r - 1}.$$

This derivation of the formula is attractive and is quite convincing. However, it is not as logically airtight as the proof by mathematical induction.

Deducing Additional Formulas

To go from one step to another in the previous calculations, the argument is made that each term among those indicated by the ellipsis (. . .) has such-and-such an appearance and when these are canceled such-and-such occurs.

But it is impossible actually to see each such term and each such calculation, and so the accuracy of these claims cannot be fully checked.

With mathematical induction it is possible to focus exactly on what happens in the middle of the ellipsis and verify without doubt that the calculations are correct.