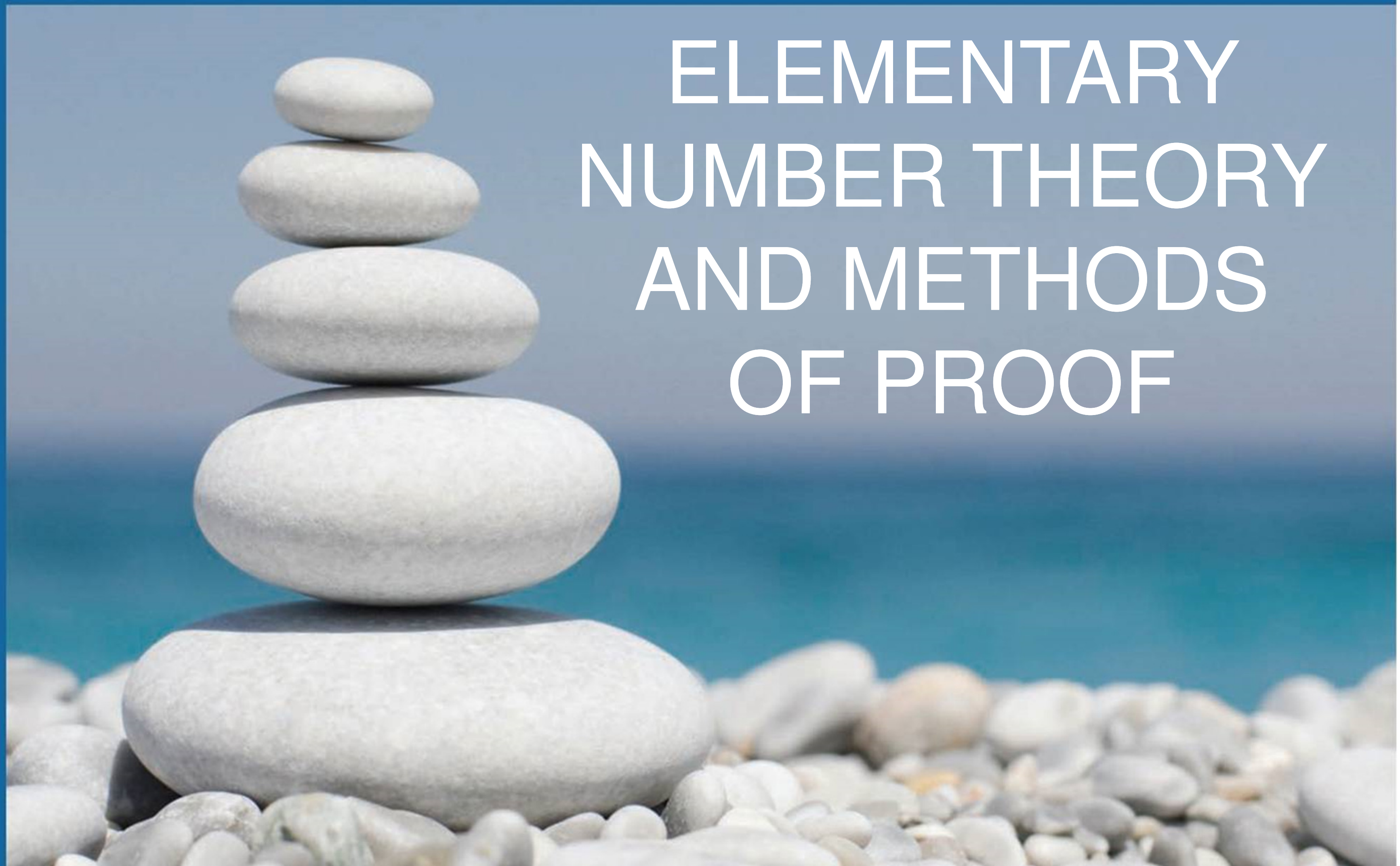
CHAPTER 4

ELEMENTARY NUMBER THEORY AND METHODS OF PROOF



SECTION 4.5

Direct Proof and Counterexample V: Floor and Ceiling

Direct Proof and Counterexample V: Floor and Ceiling

Imagine a real number sitting on a number line. The *floor* and *ceiling* of the number are the integers to the immediate left and to the immediate right of the number (unless the number is, itself, an integer, in which case its floor and ceiling both equal the number itself).

Many computer languages have built-in functions that compute floor and ceiling automatically. These functions are very convenient to use when writing certain kinds of computer programs.

In addition, the concepts of floor and ceiling are important in analyzing the efficiency of many computer algorithms.

Direct Proof and Counterexample V: Floor and Ceiling

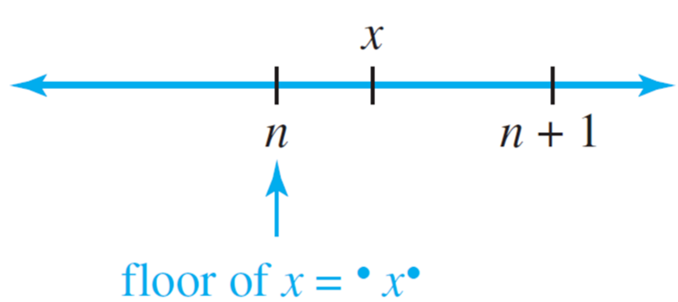
• Definition

Given any real number x, the **floor of x,** denoted $\lfloor x \rfloor$, is defined as follows:

$$\lfloor x \rfloor = \text{that unique integer } n \text{ such that } n \leq x < n + 1.$$

Symbolically, if x is a real number and n is an integer, then

$$\lfloor x \rfloor = n \quad \Leftrightarrow \quad n \leq x < n + 1.$$



Direct Proof and Counterexample V: Floor and Ceiling

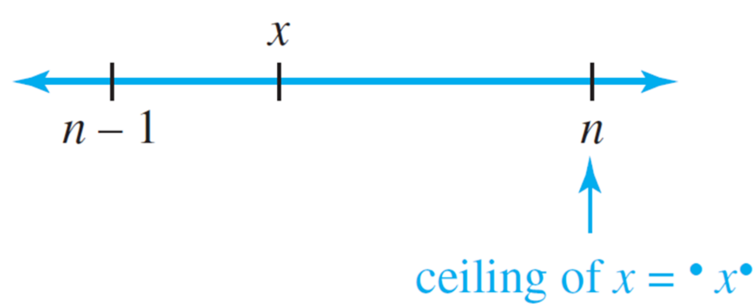
• Definition

Given any real number x, the **ceiling of x,** denoted $\lceil x \rceil$, is defined as follows:

$$\lceil x \rceil = \text{that unique integer } n \text{ such that } n - 1 < x \leq n.$$

Symbolically, if x is a real number and n is an integer, then

$$\lceil x \rceil = n \quad \Leftrightarrow \quad n - 1 < x \leq n.$$



Example 1 – *Computing Floors and Ceilings*

Compute $\lfloor x \rfloor$ and $\lceil x \rceil$ for each of the following values of *x*:

a. 25/4 **b.** 0.999 **c.** –2.01

Solution:

a. $25/4 = 6.25$ and $6 < 6.25 < 7$; hence $\lfloor 25/4 \rfloor = 6$ and $\lceil 25/4 \rceil = 7$.

b. $0 < 0.999 < 1$; hence $\lfloor 0.999 \rfloor = 0$ and $\lceil 0.999 \rceil = 1$.

c. $-3 < -2.01 < -2$; hence $\lfloor -2.01 \rfloor = -3$ and $\lceil -2.01 \rceil = -2$.

Note that on some calculators $\lfloor x \rfloor$ is denoted INT (*x*).

Example 4 – *Disproving an Alleged Property of Floor*

Is the following statement true or false?

For all real numbers *x* and *y*, $\lfloor x + y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor$.

Solution:

The statement is false. As a counterexample, take $x = y = \frac{1}{2}$.

Then

$$\lfloor x \rfloor + \lfloor y \rfloor = \left\lfloor \frac{1}{2} \right\rfloor + \left\lfloor \frac{1}{2} \right\rfloor = 0 + 0 = 0,$$

whereas

$$\lfloor x + y \rfloor = \left\lfloor \frac{1}{2} + \frac{1}{2} \right\rfloor = \lfloor 1 \rfloor = 1.$$

Example 4 – *Solution*

cont'd

Hence $\lfloor x + y \rfloor \neq \lfloor x \rfloor + \lfloor y \rfloor$.

To arrive at this counterexample, you could have reasoned as follows: Suppose *x* and *y* are real numbers. Must it necessarily be the case that $\lfloor x + y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor$, or could *x* and *y* be such that $\lfloor x + y \rfloor \neq \lfloor x \rfloor + \lfloor y \rfloor$?

Imagine values that the various quantities could take.

Example 4 – *Solution*

cont'd

For instance, if both *x* and *y* are positive, then $\lfloor x \rfloor$ and $\lfloor y \rfloor$ are the integer parts of $\lfloor x \rfloor$ and $\lfloor y \rfloor$ respectively; just as

$$2\frac{3}{5} = 2 + \frac{3}{5}$$

integer part fractional part

so is

$$x = \lfloor x \rfloor + \text{fractional part of } x$$

and

$$y = \lfloor y \rfloor + \text{fractional part of } y.$$

Where the term *fractional part* is understood here to mean the part of the number to the right of the decimal point when the number is written in decimal notation.

Example 4 – *Solution*

cont'd

Thus if *x* and *y* are positive,

$$x + y = \lfloor x \rfloor + \lfloor y \rfloor + \textit{the sum of the fractional parts of } x \textit{ and } y.$$

But also

$$x + y = \lfloor x + y \rfloor + \textit{the fractional part of } (x + y).$$

These equations show that if there exist numbers *x* and *y* such that the sum of the fractional parts of *x* and *y* is at least 1, then a counterexample can be found.

Example 4 – *Solution*

cont'd

But there do exist such x and y; for instance, $x = \frac{1}{2}$ and $y = \frac{1}{2}$ as before.

Direct Proof and Counterexample V: Floor and Ceiling

The analysis of Example 4 indicates that if *x* and *y* are positive and the sum of their fractional parts is less than 1, then $\lfloor x + y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor$.

In particular, if *x* is positive and *m* is a positive integer, then $\lfloor x + m \rfloor = \lfloor x \rfloor + \lfloor m \rfloor = \lfloor x \rfloor + m$. (The fractional part of *m* is 0; hence the sum of the fractional parts of *x* and *m* equals the fractional part of *x*, which is less than 1.)

It turns out that you can use the definition of floor to show that this equation holds for all real numbers *x* and for all integers *m*.

Example 5 – *Proving a Property of Floor*

Prove that for all real numbers *x* and for all integers *m*, $\lfloor x + m \rfloor = \lfloor x \rfloor + m.$

Solution:

Begin by supposing that *x* is a particular but arbitrarily chosen real number and that *m* is a particular but arbitrarily chosen integer. You must show that $\lfloor x + m \rfloor = \lfloor x \rfloor + m.$

Since this is an equation involving $\lfloor x \rfloor$ and $\lfloor x + m \rfloor$, it is reasonable to give one of these quantities a name:
Let $n = \lfloor x \rfloor.$

By definition of floor,

$$n \text{ is an integer} \quad \text{and} \quad n \le x < n + 1.$$

Example 5 – *Solution*

cont'd

This double inequality enables you to compute the value of $\lfloor x + m \rfloor$ in terms of *n* by adding *m* to all sides:

$$n + m \le x + m < n + m + 1.$$

Thus the left-hand side of the equation to be shown is

$$\lfloor x + m \rfloor = n + m.$$

On the other hand, since $n = \lfloor x \rfloor$, the right-hand side of the equation to be shown is

$$\lfloor x \rfloor + m = n + m$$

also. Thus

$$\lfloor x + m \rfloor = \lfloor x \rfloor + m.$$

Example 5 – *Solution*

cont'd

This discussion is summarized as follows:

Theorem 4.5.1

For all real numbers x and all integers m, $\lfloor x + m \rfloor = \lfloor x \rfloor + m$.

Proof:

Suppose a real number x and an integer m are given. *[We must show that* $\lfloor x + m \rfloor = \lfloor x \rfloor + m$*.]*

Let $n = \lfloor x \rfloor$. By definition of floor, n is an integer and

$$n \leq x < n + 1.$$

Example 5 – *Solution*

cont'd

Add *m* to all three parts to obtain

$$n + m \leq x + m < n + m + 1$$

[since adding a number to both sides of an inequality does not change the direction of the inequality].

Now $n + m$ is an integer *[since n and m are integers and a sum of integers is an integer]*, and so, by definition of floor, the left-hand side of the equation to be shown is

$$\lfloor x + m \rfloor = n + m.$$

Example 5 – *Solution*

cont'd

But $n = \lfloor x \rfloor$. Hence, by substitution,

$$n + m = \lfloor x \rfloor + m,$$

which is the right-hand side of the equation to be shown.

Thus $\lfloor x + m \rfloor = \lfloor x \rfloor + m$ *[as was to be shown]*.

Direct Proof and Counterexample V: Floor and Ceiling

The analysis of a number of computer algorithms, such as the binary search and merge sort algorithms, requires that you know the value of $\lfloor n/2 \rfloor$, where *n* is an integer.

The formula for computing this value depends on whether *n* is even or odd.

Theorem 4.5.2 The Floor of $n/2$

For any integer n,

$$\left\lfloor \frac{n}{2} \right\rfloor = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{n-1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

Direct Proof and Counterexample V: Floor and Ceiling

Given any integer *n* and a positive integer *d*, the quotient-remainder theorem guarantees the existence of unique integers *q* and *r* such that

$$n = dq + r \quad \text{and} \quad 0 \le r < d.$$

The following theorem states that the floor notation can be used to describe *q* and *r* as follows:

$$q = \left\lfloor \frac{n}{d} \right\rfloor \quad \text{and} \quad r = n - d \left\lfloor \frac{n}{d} \right\rfloor.$$

Direct Proof and Counterexample V: Floor and Ceiling

Thus if, on a calculator or in a computer language, floor is built in but *div* and *mod* are not, *div* and *mod* can be defined as follows: For a nonnegative integer *n* and a positive integer *d*,

$$n \; div \; d = \left\lfloor \frac{n}{d} \right\rfloor \quad \text{and} \quad n \; mod \; d = n - d\left\lfloor \frac{n}{d} \right\rfloor. \qquad 4.5.1$$

Note that *d* divides *n* if, and only if, *n mod d* = 0, or, in other words, $n = d\lfloor n/d \rfloor$.

Theorem 4.5.3

If n is any integer and d is a positive integer, and if $q = \lfloor n/d \rfloor$ and $r = n - d\lfloor n/d \rfloor$, then

$$n = dq + r \quad \text{and} \quad 0 \le r < d.$$

Example 6 – *Computing div and mod*

Use the floor notation to compute 3850 *div* 17 and 3850 *mod* 17.

Solution:

By formula (4.5.1),

$$n \; div \; d = \left\lfloor \frac{n}{d} \right\rfloor \quad \text{and} \quad n \; mod \; d = n - d \left\lfloor \frac{n}{d} \right\rfloor .$$

$$\begin{aligned} 3850 \; div \; 17 &= \lfloor 3850/17 \rfloor \\ &= \lfloor 226.4705882\ldots \rfloor \\ &= 226 \end{aligned}$$

Example 6 – *Solution*

cont'd

$$
\begin{aligned}
3850 \; mod \; 17 &= 3850 - 17 \cdot \lfloor 3850/17 \rfloor \\
&= 3850 - 17 \cdot 226 \\
&= 3850 - 3842 \\
&= 8.
\end{aligned}
$$