# CHAPTER 3

# THE LOGIC OF QUANTIFIED STATEMENTS

## SECTION 3.4

# Arguments with Quantified Statements

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The rule of *universal instantiation* (in-stan-she-AY-shun) says the following:

> If some property is true of *everything* in a set, then it is true of *any particular* thing in the set.

Universal instantiation is *the* fundamental tool of deductive reasoning.

Mathematical formulas, definitions, and theorems are like general templates that are used over and over in a wide variety of particular situations.

# Arguments with Quantified Statements

A given theorem says that such and such is true for all things of a certain type.

If, in a given situation, you have a particular object of that type, then by universal instantiation, you conclude that such and such is true for that particular object.

You may repeat this process 10, 20, or more times in a single proof or problem solution.

# Universal Modus Ponens

# Universal Modus Ponens

The rule of universal instantiation can be combined with modus ponens to obtain the valid form of argument called *universal modus ponens*.

**Universal Modus Ponens**

| *Formal Version* | *Informal Version* |
|---|---|
| $\forall x$, if $P(x)$ then $Q(x)$. | If $x$ makes $P(x)$ true, then $x$ makes $Q(x)$ true. |
| $P(a)$ for a particular $a$. | $a$ makes $P(x)$ true. |
| • $Q(a)$. | • $a$ makes $Q(x)$ true. |

# Universal Modus Ponens

Note that the first, or major, premise of universal modus ponens could be written "All things that make $P(x)$ true make $Q(x)$ true," in which case the conclusion would follow by universal instantiation alone.

However, the if-then form is more natural to use in the majority of mathematical situations.

## Example 1 – *Recognizing Universal Modus Ponens*

Rewrite the following argument using quantifiers, variables, and predicate symbols. Is this argument valid? Why?

If an integer is even, then its square is even.

*k* is a particular integer that is even.

• $k^2$ is even.

Solution:

The major premise of this argument can be rewritten as

$\forall x$, if $x$ is an even integer then $x^2$ is even.

# Example 1 – *Solution*

cont'd

Let $E(x)$ be "$x$ is an even integer," let $S(x)$ be "$x^2$ is even," and let $k$ stand for a particular integer that is even.

Then the argument has the following form:

$\forall x$, if $E(x)$ then $S(x)$.

$E(k)$, for a particular $k$.

• $S(k)$.

This argument has the form of universal modus ponens and is therefore valid.

# Use of Universal Modus Ponens in a Proof

# Use of Universal Modus Ponens in a Proof

Here is a proof that the sum of any two even integers is even.

It makes use of the definition of even integer, namely, that an integer is *even* if, and only if, it equals twice some integer. (Or, more formally: $\forall$ integers $x$, $x$ is even if, and only if, $\exists$ an integer $k$ such that $x = 2k$.)

Suppose $m$ and $n$ are particular but arbitrarily chosen even integers. Then $m = 2r$ for some integer $r$,[(1)] and $n = 2s$ for some integer $s$.[(2)]

# Use of Universal Modus Ponens in a Proof

Hence

$$m + n = 2r + 2s \qquad \text{by substitution}$$
$$= 2(r + s)^{(3)} \qquad \text{by factoring out the 2.}$$

Now $r + s$ is an integer,[(4)] and so $2(r + s)$ is even.[(5)]

Thus $m + n$ is even.

# Use of Universal Modus Ponens in a Proof

The following expansion of the proof shows how each of the numbered steps is justified by arguments that are valid by universal modus ponens.

(1) If an integer is even, then it equals twice some integer.

*m* is a particular even integer.

- *m* equals twice some integer *r*.

(2) If an integer is even, then it equals twice some integer.

*n* is a particular even integer.

- *n* equals twice some integer *s*.

# Use of Universal Modus Ponens in a Proof

(3) If a quantity is an integer, then it is a real number.

$r$ and $s$ are particular integers.

- $r$ and $s$ are real numbers.

For all $a$, $b$, and $c$, if $a$, $b$, and $c$ are real numbers, then $ab + ac = a(b + c)$.

2, $r$, and $s$ are particular real numbers.

- $2r + 2s = 2(r + s)$.

(4) For all $u$ and $v$, if $u$ and $v$ are integers, then $u + v$ is an integer.

$r$ and $s$ are two particular integers.

- $r + s$ is an integer.

# Use of Universal Modus Ponens in a Proof

(5) If a number equals twice some integer, then that number is even.

$2(r + s)$ equals twice the integer $r + s$.

• $2(r + s)$ is even.

## Universal Modus Tollens

# Universal Modus Tollens

Another crucially important rule of inference is *universal modus tollens*. Its validity results from combining universal instantiation with modus tollens.

Universal modus tollens is the heart of proof of contradiction, which is one of the most important methods of mathematical argument.

**Universal Modus Tollens**

| *Formal Version* | *Informal Version* |
|---|---|
| $\forall x$, if $P(x)$ then $Q(x)$. | If $x$ makes $P(x)$ true, then $x$ makes $Q(x)$ true. |
| $\sim Q(a)$, for a particular $a$. | $a$ does not make $Q(x)$ true. |
| • $\sim P(a)$. | • $a$ does not make $P(x)$ true. |

## Example 3 – *Recognizing the Form of Universal Modus Tollens*

Rewrite the following argument using quantifiers, variables, and predicate symbols. Write the major premise in conditional form. Is this argument valid? Why?

All human beings are mortal.

Zeus is not mortal.

• Zeus is not human.

Solution:

The major premise can be rewritten as

$\forall x$, if $x$ is human then $x$ is mortal.

# Example 3 – *Solution*

cont'd

Let $H(x)$ be "$x$ is human," let $M(x)$ be "$x$ is mortal," and let $Z$ stand for Zeus.

The argument becomes

$$\forall x, \text{ if } H(x) \text{ then } M(x)$$

$$\sim M(Z)$$

$$\bullet \quad \sim H(Z).$$

This argument has the form of universal modus tollens and is therefore valid.

# Proving Validity of Arguments with Quantified Statements

# Proving Validity of Arguments with Quantified Statements

The intuitive definition of validity for arguments with quantified statements is the same as for arguments with compound statements.

An argument is valid if, and only if, the truth of its conclusion follows *necessarily* from the truth of its premises.

The formal definition is as follows:

> **• Definition**
>
> To say that an *argument form* is **valid** means the following: No matter what particular predicates are substituted for the predicate symbols in its premises, if the resulting premise statements are all true, then the conclusion is also true. An *argument* is called **valid** if, and only if, its form is valid.

# Using Diagrams to Test for Validity

# Using Diagrams to Test for Validity

Consider the statement

All integers are rational numbers.

Or, formally,

$\forall$ integers *n*, *n* is a rational number.

Picture the set of all integers and the set of all rational numbers as disks.

# Using Diagrams to Test for Validity

The truth of the given statement is represented by placing the integers disk entirely inside the rationals disk, as shown in Figure 3.4.1.

Because the two statements "$\forall x \in D, Q(x)$" and "$\forall x$, if $x$ is in $D$ then $Q(x)$" are logically equivalent, both can be represented by diagrams like the foregoing.

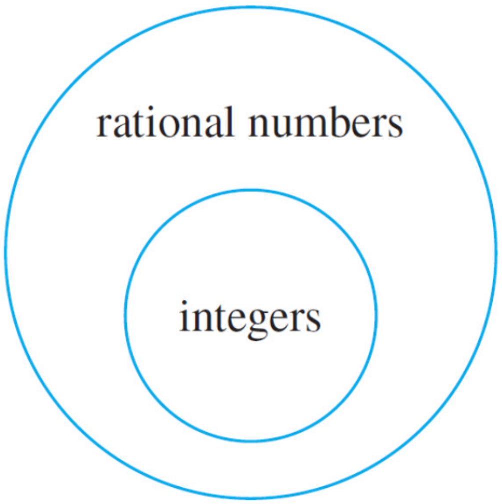


**Figure 3.4.1**

# Using Diagrams to Test for Validity

To test the validity of an argument diagrammatically, represent the truth of both premises with diagrams.

Then analyze the diagrams to see whether they necessarily represent the truth of the conclusion as well.

# Example 6 – *Using Diagrams to Show Invalidity*

Use a diagram to show the invalidity of the following argument:

All human beings are mortal.

Felix is mortal.

• Felix is a human being.

# Example 6 – *Solution*

The major and minor premises are represented diagrammatically in Figure 3.4.4.

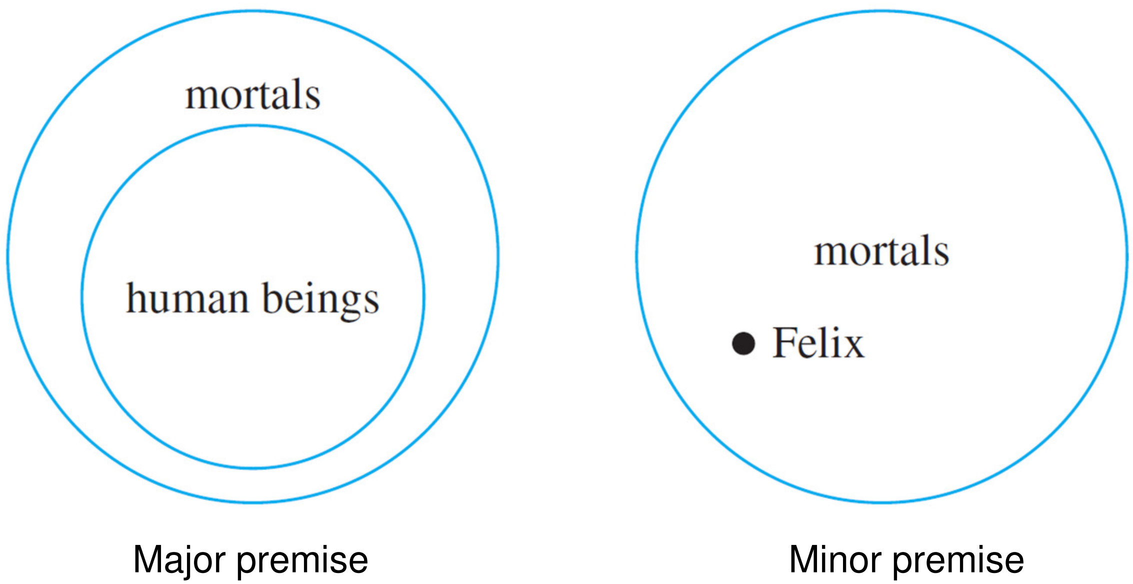


**Figure 3.4.4**

# Example 6 – *Solution*

cont'd

All that is known is that the Felix dot is located *somewhere* inside the mortals disk. Where it is located with respect to the human beings disk cannot be determined. Either one of the situations shown in Figure 3.4.5 might be the case.

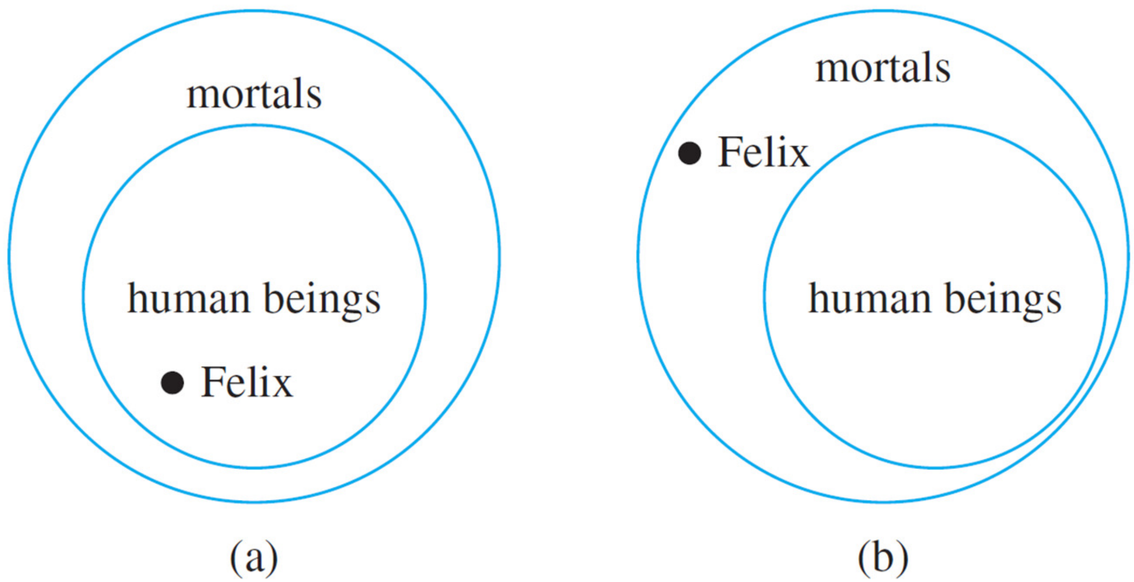


**Figure 3.4.5**

# Example 6 – *Solution*

cont'd

The conclusion "Felix is a human being" is true in the first case but not in the second (Felix might, for example, be a cat).

Because the conclusion does not necessarily follow from the premises, the argument is invalid.

# Using Diagrams to Test for Validity

The argument of Example 6 would be valid if the major premise were replaced by its converse. But since a universal conditional statement is not logically equivalent to its converse, such a replacement cannot, in general, be made.

We say that this argument exhibits the converse error.

**Converse Error (Quantified Form)**

| *Formal Version* | *Informal Version* |
| --- | --- |
| $\forall x$, if $P(x)$ then $Q(x)$. | If $x$ makes $P(x)$ true, then $x$ makes $Q(x)$ true. |
| $Q(a)$ for a particular $a$. | $a$ makes $Q(x)$ true. |
| • $P(a)$. ← invalid conclusion | • $a$ makes $P(x)$ true. ← invalid conclusion |

# Using Diagrams to Test for Validity

The following form of argument would be valid if a conditional statement were logically equivalent to its inverse. But it is not, and the argument form is invalid.

We say that it exhibits the inverse error.

**Inverse Error (Quantified Form)**

| *Formal Version* | *Informal Version* |
| --- | --- |
| $\forall x$, if $P(x)$ then $Q(x)$. | If $x$ makes $P(x)$ true, then $x$ makes $Q(x)$ true. |
| $\sim P(a)$, for a particular $a$. | $a$ does not make $P(x)$ true. |
| • $\sim Q(a)$. ← invalid conclusion | • $a$ does not make $Q(x)$ true. ← invalid conclusion |

# Example 7 – *An Argument with "No"*

Use diagrams to test the following argument for validity:

No polynomial functions have horizontal asymptotes.

This function has a horizontal asymptote.

- This function is not a polynomial function.

# Example 7 – *Solution*

A good way to represent the major premise diagrammatically is shown in Figure 3.4.6, two disks—a disk for polynomial functions and a disk for functions with horizontal asymptotes—that do not overlap at all.

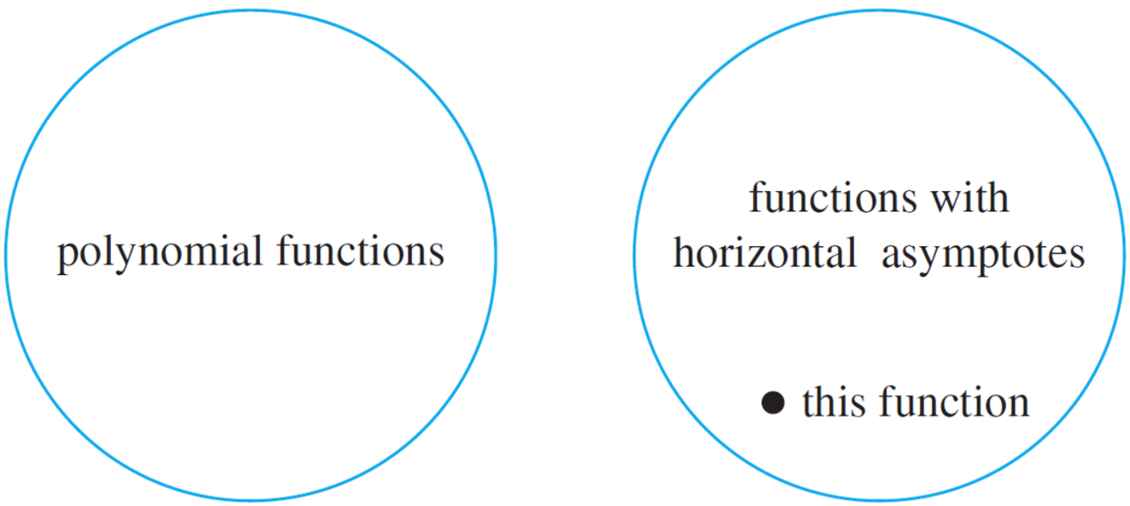


Figure 3.4.6

# Example 7 – *Solution*

cont'd

The minor premise is represented by placing a dot labeled "this function" inside the disk for functions with horizontal asymptotes.

The diagram shows that "this function" must lie outside the polynomial functions disk, and so the truth of the conclusion necessarily follows from the truth of the premises.

Hence the argument is valid.

# Using Diagrams to Test for Validity

An alternative approach to this example is to transform the statement "No polynomial functions have horizontal asymptotes" into the equivalent form "$\forall x$, if $x$ is a polynomial function, then $x$ does not have a horizontal asymptote."

# Using Diagrams to Test for Validity

If this is done, the argument can be seen to have the form

$$\forall x, \text{ if } P(x) \text{ then } Q(x).$$

$$\sim Q(a), \text{ for a particular } a.$$

$$\bullet \; \sim P(a).$$

where $P(x)$ is "$x$ is a polynomial function" and $Q(x)$ is "$x$ does not have a horizontal asymptote."

This is valid by universal modus tollens.

# Creating Additional Forms of Argument

# Creating Additional Forms of Argument

Universal modus ponens and modus tollens were obtained by combining universal instantiation with modus ponens and modus tollens.

In the same way, additional forms of arguments involving universally quantified statements can be obtained by combining universal instantiation with other of the valid argument forms discussed earlier.

# Creating Additional Forms of Argument

Consider the following argument:

$$p \to q$$
$$q \to r$$
$$\bullet \quad p \to r$$

This argument form can be combined with universal instantiation to obtain the following valid argument form.

**Universal Transitivity**

| *Formal Version* | *Informal Version* |
|---|---|
| $\forall x\, P(x) \to Q(x).$ | Any $x$ that makes $P(x)$ true makes $Q(x)$ true. |
| $\forall x\, Q(x) \to R(x).$ | Any $x$ that makes $Q(x)$ true makes $R(x)$ true. |
| • $\forall x\, P(x) \to R(x).$ | • Any $x$ that makes $P(x)$ true makes $R(x)$ true. |

# Example 8 – *Evaluating an Argument for Tarski's World*

Consider the Tarski world shown in Figure 3.3.1.

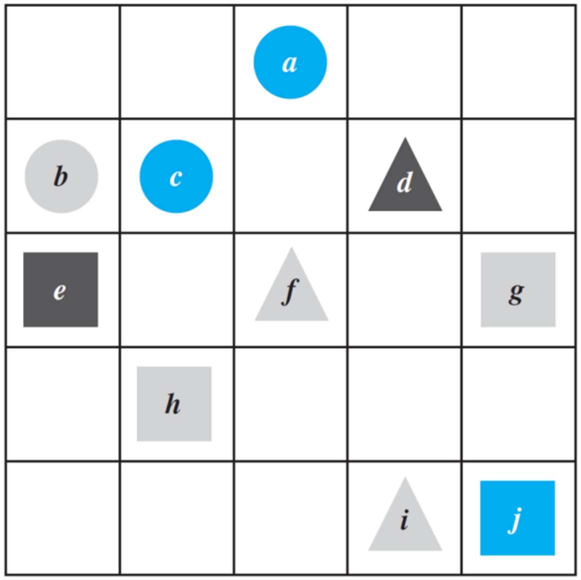


**Figure 3.3.1**

# Example 8 – *Evaluating an Argument for Tarski's World*

cont'd

Reorder and rewrite the premises to show that the conclusion follows as a valid consequence from the premises.

1. All the triangles are blue.

2. If an object is to the right of all the squares, then it is above all the circles.

3. If an object is not to the right of all the squares, then it is not blue.

- All the triangles are above all the circles.

# Example 8 – *Solution*

It is helpful to begin by rewriting the premises and the conclusion in if-then form:

1. $\forall x$, if $x$ is a triangle, then $x$ is blue.

2. $\forall x$, if $x$ is to the right of all the squares, then $x$ is above all the circles.

3. $\forall x$, if $x$ is not to the right of all the squares, then $x$ is not blue.

- $\forall x$, if $x$ is a triangle, then $x$ is above all the circles.

# Example 8 – *Solution*

cont'd

The goal is to reorder the premises so that the conclusion of each is the same as the hypothesis of the next.

Also, the hypothesis of the argument's conclusion should be the same as the hypothesis of the first premise, and the conclusion of the argument's conclusion should be the same as the conclusion of the last premise.

To achieve this goal, it may be necessary to rewrite some of the statements in contrapositive form.

# Example 8 – *Solution*

cont'd

In this example you can see that the first premise should remain where it is, but the second and third premises should be interchanged.

Then the hypothesis of the argument is the same as the hypothesis of the first premise, and the conclusion of the argument's conclusion is the same as the conclusion of the third premise.

But the hypotheses and conclusions of the premises do not quite line up. This is remedied by rewriting the third premise in contrapositive form.

# Example 8 – *Solution*

cont'd

Thus the premises and conclusion of the argument can be rewritten as follows:

1. $\forall x$, if $x$ is a triangle, then $x$ is blue.

3. $\forall x$, if $x$ is blue, then $x$ is to the right of all the squares.

2. $\forall x$, if $x$ is to the right of all the squares, then $x$ is above all the circles.

• $\forall x$, if $x$ is a triangle, then $x$ is above all the circles.

# Example 8 – *Solution*

cont'd

The validity of this argument follows easily from the validity of universal transitivity.

Putting 1 and 3 together and using universal transitivity gives that

4. $\forall x$, if $x$ is a triangle, then $x$ is to the right of all the squares.

And putting 4 together with 2 and using universal transitivity gives that

$\forall x$, if $x$ is a triangle, then $x$ is above all the circles,

which is the conclusion of the argument.

# Remark on the Converse and Inverse Errors

# Remark on the Converse and Inverse Errors

A variation of the converse error is a very useful reasoning tool, provided that it is used with caution.

It is the type of reasoning that is used by doctors to make medical diagnoses and by auto mechanics to repair cars.

# Remark on the Converse and Inverse Errors

It is the type of reasoning used to generate explanations for phenomena. It goes like this: If a statement of the form

For all $x$, if $P(x)$ then $Q(x)$

is true, and if

$Q(a)$ is true, for a particular $a$,

then check out the statement $P(a)$; it just might be true.

# Remark on the Converse and Inverse Errors

For instance, suppose a doctor knows that

> For all *x*, if *x* has pneumonia, then *x* has a fever and chills, coughs deeply, and feels exceptionally tired and miserable.

And suppose the doctor also knows that

> John has a fever and chills, coughs deeply,
> and feels exceptionally tired and miserable.

On the basis of these data, the doctor concludes that a diagnosis of pneumonia is a strong possibility, though not a certainty.

# Remark on the Converse and Inverse Errors

The doctor will probably attempt to gain further support for this diagnosis through laboratory testing that is specifically designed to detect pneumonia.

Note that the closer a set of symptoms comes to being a necessary and sufficient condition for an illness, the more nearly certain the doctor can be of his or her diagnosis.

This form of reasoning has been named **abduction** by researchers working in artificial intelligence. It is used in certain computer programs, called expert systems, that attempt to duplicate the functioning of an expert in some field of knowledge.