CHAPTER 3

THE LOGIC OF QUANTIFIED STATEMENTS

SECTION 3.3

Statements with Multiple Quantifiers

Statements with Multiple Quantifiers

When a statement contains more than one quantifier, we imagine the actions suggested by the quantifiers as being performed in the order in which the quantifiers occur.

For instance, consider a statement of the form

$\forall x$ in set D, $\exists y$ in set E such that x and y satisfy property $P(x, y)$.

Statements with Multiple Quantifiers

To show that such a statement is true, you must be able to meet the following challenge:

•Imagine that someone is allowed to choose any element whatsoever from the set D, and imagine that the person gives you that element. Call it x.

•The challenge for you is to find an element y in E so that the person's x and your y, taken together, satisfy property $P(x, y)$.

Example 1 – *Truth of a ∀∃ Statement in a Tarski World*

Consider the Tarski world shown in Figure 3.3.1.

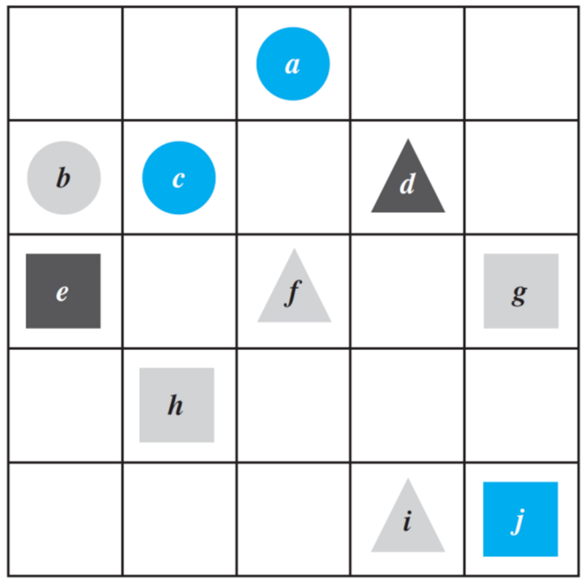


Figure 3.3.1

Show that the following statement is true in this world: For all triangles *x*, there is a square *y* such that *x* and *y* have the same color.

Example 1 – *Solution*

The statement says that no matter which triangle someone gives you, you will be able to find a square of the same color. There are only three triangles, *d*, *f*, and *i*.

The following table shows that for each of these triangles a square of the same color can be found.

Given x =	**choose y =**	**and check that y is the same color as x.**
d	e	yes •
f or i	h or g	yes •

Statements with Multiple Quantifiers

Now consider a statement containing both ∀ and ∃, where the ∃ comes before the ∀:

∃ an x in D such that ∀ y in E, x and y satisfy property $P(x, y)$.

To show that a statement of this form is true:
You must find one single element (call it x) in D with the following property:

•After you have found your x, someone is allowed to choose any element whatsoever from E. The person challenges you by giving you that element. Call it y.

•Your job is to show that your x together with the person's y satisfy property $P(x, y)$.

Statements with Multiple Quantifiers

Here is a summary of the convention for interpreting statements with two different quantifiers:

Interpreting Statements with Two Different Quantifiers

If you want to establish the truth of a statement of the form

$$\forall x \text{ in } D, \exists y \text{ in } E \text{ such that } P(x, y)$$

your challenge is to allow someone else to pick whatever element *x* in *D* they wish and then you must find an element *y* in *E* that "works" for that particular *x*.

Statements with Multiple Quantifiers

If you want to establish the truth of a statement of the form

$$\exists x \text{ in } D \text{ such that } \forall y \text{ in } E,\ P(x, y)$$

your job is to find one particular x in D that will "work" no matter what y in E anyone might choose to challenge you with.

Example 3 – *Interpreting Multiply-Quantified Statements*

A college cafeteria line has four stations: salads, main courses, desserts, and beverages.

The salad station offers a choice of green salad or fruit salad; the main course station offers spaghetti or fish; the dessert station offers pie or cake; and the beverage station offers milk, soda, or coffee. Three students, Uta, Tim, and Yuen, go through the line and make the following choices:

Uta: green salad, spaghetti, pie, milk

Tim: fruit salad, fish, pie, cake, milk, coffee

Yuen: spaghetti, fish, pie, soda

Example 3 – *Interpreting Multiply-Quantified Statements*
cont'd

These choices are illustrated in Figure 3.3.2.

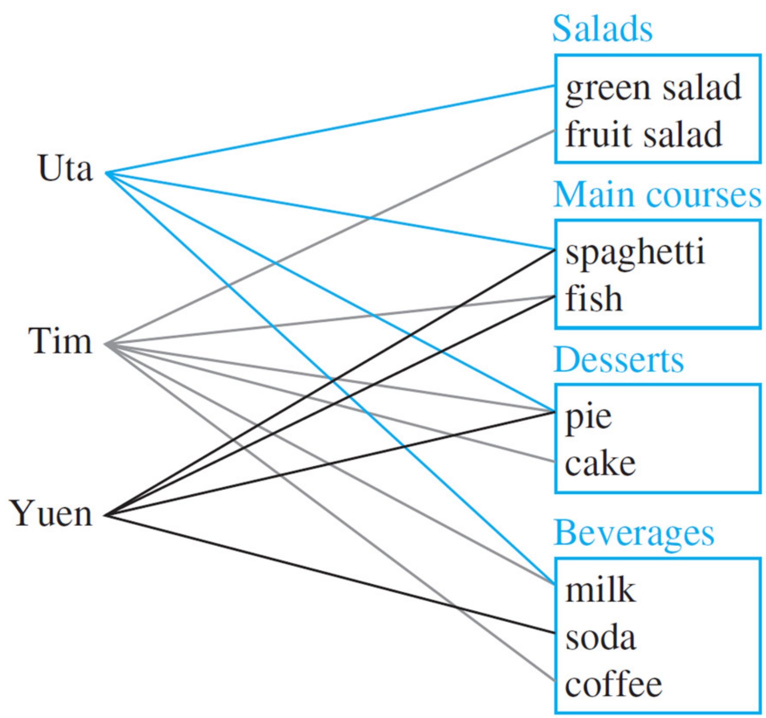


Figure 3.3.2

Example 3 – *Interpreting Multiply-Quantified Statements* cont'd

Write each of following statements informally and find its truth value.

a. $\exists$ an item I such that $\forall$ students S, S chose I.

b. $\exists$ a student S such that $\forall$ items I, S chose I.

c. $\exists$ a student S such that $\forall$ stations Z, $\exists$ an item I in Z such that S chose I.

d. $\forall$ students S and $\forall$ stations Z, $\exists$ an item I in Z such that S chose I.

Example 3 – *Solution*

a. There is an item that was chosen by every student. This is true; every student chose pie.

b. There is a student who chose every available item. This is false; no student chose all nine items.

c. There is a student who chose at least one item from every station. This is true; both Uta and Tim chose at least one item from every station.

d. Every student chose at least one item from every station. This is false; Yuen did not choose a salad.

Translating from Informal to Formal Language

Translating from Informal to Formal Language

Most problems are stated in informal language, but solving them often requires translating them into more formal terms.

Example 4 – *Translating Multiply-Quantified Statements from Informal to Formal Language*

The **reciprocal** of a real number a is a real number b such that $ab = 1$. The following two statements are true. Rewrite them formally using quantifiers and variables:

a. Every nonzero real number has a reciprocal.

b. There is a real number with no reciprocal.

The number 0 has no reciprocal.

Solution:

a. $\forall$ nonzero real numbers u, $\exists$ a real number v such that $uv = 1$.

b. $\exists$ a real number c such that $\forall$ real numbers d, $cd \neq 1$.

Ambiguous Language

Ambiguous Language

Imagine you are visiting a factory that manufactures computer microchips. The factory guide tells you,

> There is a person supervising every detail of the production process.

Note that this statement contains informal versions of both the existential quantifier *there is* and the universal quantifier *every*.

Ambiguous Language

Which of the following best describes its meaning?

- There is one single person who supervises all the details of the production process.
- For any particular production detail, there is a person who supervises that detail, but there might be different supervisors for different details.

Ambiguous Language

Once you interpreted the sentence in one way, it may have been hard for you to see that it could be understood in the other way.

Perhaps you had difficulty even though the two possible meanings were explained.

Although statements written informally may be open to multiple interpretations, we cannot determine their truth or falsity without interpreting them one way or another.

Therefore, we have to use context to try to ascertain their meaning as best we can.

Negations of Multiply-Quantified Statements

Negations of Multiply-Quantified Statements

You can use the same rules to negate multiply-quantified statements that you used to negate simpler quantified statements.

We have known that

$$\sim(\forall x \text{ in } D,\ P(x)) \equiv \exists x \text{ in } D \text{ such that } \sim P(x).$$

and

$$\sim(\exists x \text{ in } D \text{ such that } P(x)) \equiv \forall x \text{ in } D, \sim P(x).$$

Negations of Multiply-Quantified Statements

We apply these laws to find

$$\sim(\forall x \text{ in } D, \exists y \text{ in } E \text{ such that } P(x, y))$$

by moving in stages from left to right along the sentence.

First version of negation: $\exists x$ in D such that $\sim(\exists y$ in E such that $P(x, y))$.

Final version of negation: $\exists x$ in D such that $\forall y$ in E, $\sim P(x, y)$.

Negations of Multiply-Quantified Statements

Similarly, to find

$$\sim(\exists x \text{ in } D \text{ such that } \forall y \text{ in } E,\ P(x, y)),$$

we have

First version of negation: $\forall x$ in D, $\sim(\forall y$ in E, $P(x, y))$.

Final version of negation: $\forall x$ in D, $\exists y$ in E such that $\sim P(x, y)$.

Negations of Multiply-Quantified Statements

These facts can be summarized as follows:

Negations of Multiply-Quantified Statements

$$\sim(\forall x \text{ in } D, \exists y \text{ in } E \text{ such that } P(x, y)) \equiv \exists x \text{ in } D \text{ such that } \forall y \text{ in } E, \sim P(x, y).$$

$$\sim(\exists x \text{ in } D \text{ such that } \forall y \text{ in } E, P(x, y)) \equiv \forall x \text{ in } D, \exists y \text{ in } E \text{ such that } \sim P(x, y).$$

Example 8 – *Negating Statements in a Tarski World*

Refer to the Tarski world of Figure 3.3.1.

Write a negation for each of the following statements, and determine which is true, the given statement or its negation.

a. For all squares *x*, there is a circle *y* such that *x* and *y* have the same color.

b. There is a triangle *x* such that for all squares *y*, *x* is to the right of *y*.

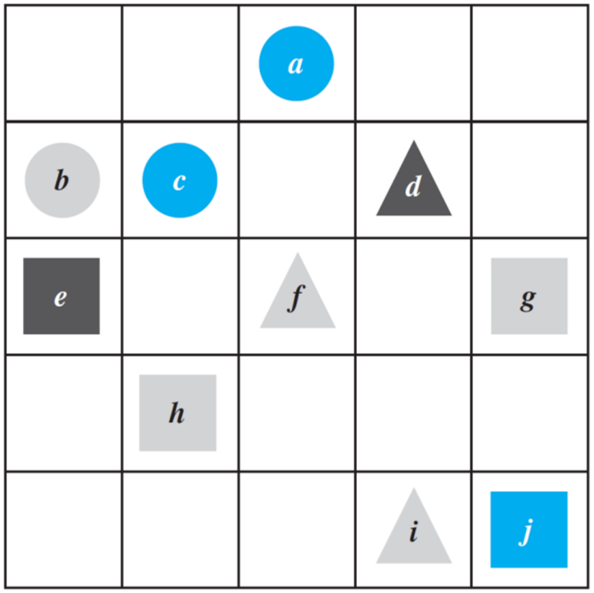


Figure 3.3.1

Example 8(a) – *Solution*

First version of negation: ∃ a square *x* such that ∼(∃ a circle *y* such that *x* and *y* have the same color).

Final version of negation: ∃ a square *x* such that ∀ circles *y*, *x* and *y* do not have the same color.

The negation is true. Square *e* is black and no circle is black, so there is a square that does not have the same color as any circle.

Example 8(b) – *Solution*

cont'd

First version of negation: ∀ triangles x,~ (∀ squares y, x is to the right of y).

Final version of negation: ∀ triangles x, ∃ a square y such that x is not to the right of y.

The negation is true because no matter what triangle is chosen, it is not to the right of square g (or square j).

Order of Quantifiers

Order of Quantifiers

Consider the following two statements:

$\forall$ people x, $\exists$ a person y such that x loves y.

$\exists$ a person y such that $\forall$ people x, x loves y.

Note that except for the order of the quantifiers, these statements are identical.

However, the first means that given any person, it is possible to find someone whom that person loves, whereas the second means that there is one amazing individual who is loved by all people.

Order of Quantifiers

The two sentences illustrate an extremely important property about multiply-quantified statements:

> In a statement containing both $\forall$ and $\exists$, changing the order of the quantifiers usually changes the meaning of the statement.

Interestingly, however, if one quantifier immediately follows another quantifier *of the same type*, then the order of the quantifiers does not affect the meaning.

Example 9 – *Quantifier Order in a Tarski World*

Look again at the Tarski world of Figure 3.3.1. Do the following two statements have the same truth value?

a. For every square *x* there is a triangle *y* such that *x* and *y* have different colors.

b. There exists a triangle *y* such that for every square *x*, *x* and *y* have different colors.

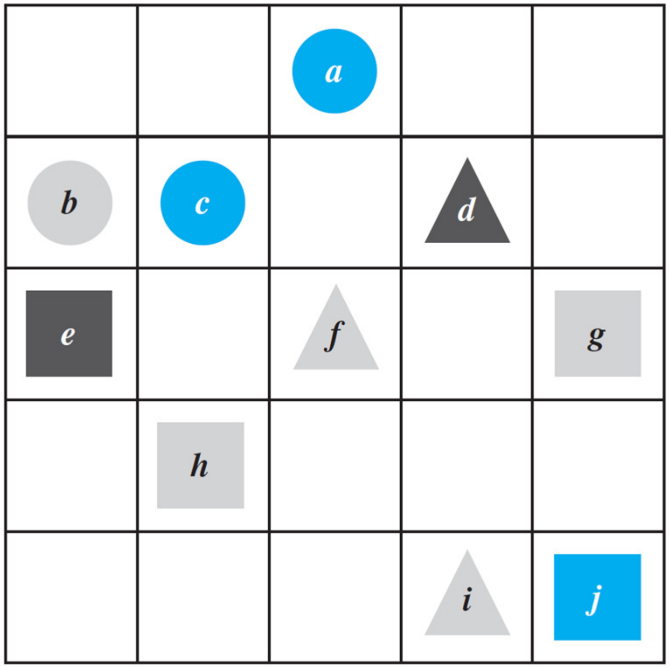


Figure 3.3.1

Example 9 – *Solution*

Statement (a) says that if someone gives you one of the squares from the Tarski world, you can find a triangle that has a different color. This is true.

If someone gives you square *g* or *h* (which are gray), you can use triangle *d* (which is black); if someone gives you square *e* (which is black), you can use either triangle *f* or triangle *i* (which are both gray); and if someone gives you square *j* (which is blue), you can use triangle *d* (which is black) or triangle *f* or *i* (which are both gray).

Example 9 – *Solution*

cont'd

Statement (b) says that there is one particular triangle in the Tarski world that has a different color from every one of the squares in the world. This is false.

Two of the triangles are gray, but they cannot be used to show the truth of the statement because the Tarski world contains gray squares.

The only other triangle is black, but it cannot be used either because there is a black square in the Tarski world.

Thus one of the statements is true and the other is false, and so they have opposite truth values.

Formal Logical Notation

Formal Logical Notation

In some areas of computer science, logical statements are expressed in purely symbolic notation.

The notation involves using predicates to describe all properties of variables and omitting the words *such that* in existential statements.

The formalism also depends on the following facts:

"$\forall x$ in D, $P(x)$" can be written as "$\forall x(x \text{ in } D \to P(x))$," and "$\exists x$ in D such that $P(x)$" can be written as "$\exists x(x \text{ in } D \wedge P(x))$."

We illustrate the use of these facts in Example 10.

Example 10 – *Formalizing Statements in a Tarski World*

Consider once more the Tarski world of Figure 3.3.1:

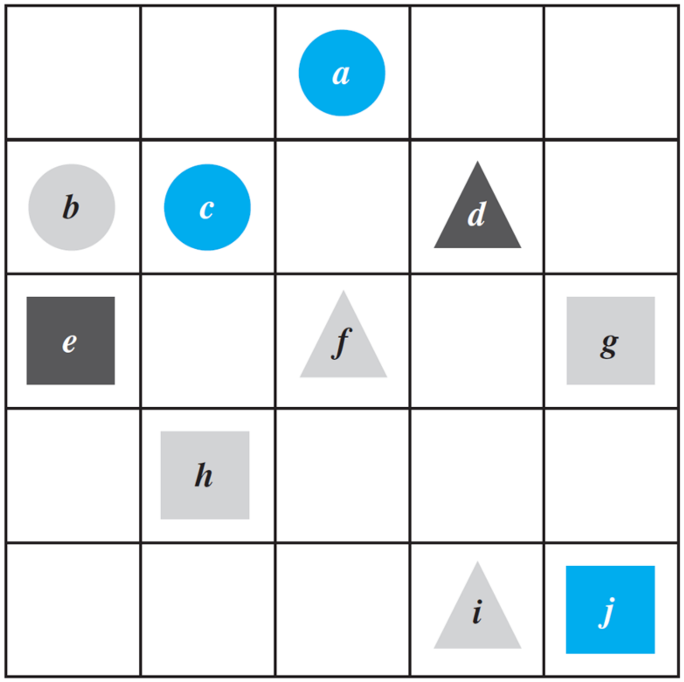


Figure 3.3.1

Example 10 – *Formalizing Statements in a Tarski World* cont'd

Let Triangle(x), Circle(x), and Square(x) mean "x is a triangle," "x is a circle," and "x is a square"; let Blue(x), Gray(x), and Black(x) mean "x is blue," "x is gray," and "x is black";

let RightOf(x, y), Above(x, y), and SameColorAs(x, y) mean "x is to the right of y," "x is above y," and "x has the same color as y"; and use the notation $x = y$ to denote the predicate "x is equal to y".

Let the common domain D of all variables be the set of all the objects in the Tarski world.

Example 10 – *Formalizing Statements in a Tarski World*

cont'd

Use formal, logical notation to write each of the following statements, and write a formal negation for each statement.

a. For all circles *x*, *x* is above *f*.

b. There is a square *x* such that *x* is black.

c. For all circles *x*, there is a square *y* such that *x* and *y* have the same color.

d. There is a square *x* such that for all triangles *y*, *x* is to right of *y*.

Example 10(a) – *Solution*

Statement:

$$\forall x(\text{Circle}(x) \rightarrow \text{Above}(x, f)).$$

Negation:

$$\sim(\forall x(\text{Circle}(x) \rightarrow \text{Above}(x, f))$$

$$\equiv \exists x \sim (\text{Circle}(x) \rightarrow \text{Above}(x, f))$$

by the law for negating a ∀ statement

$$\equiv \exists x(\text{Circle}(x) \wedge \sim\text{Above}(x, f))$$

by the law of negating an if-then statement

Example 10(b) – *Solution*

cont'd

Statement:

$$\exists x(\text{Square}(x) \wedge \text{Black}(x)).$$

Negation:

$$\sim(\exists x(\text{Square}(x) \wedge \text{Black}(x))$$

$$\equiv \forall x \sim (\text{Square}(x) \wedge \text{Black}(x))$$

by the law for negating a $\exists$ statement

$$\equiv \forall x(\sim\text{Square}(x) \vee \sim\text{Black}(x))$$

by De Morgan's law

Example 10(c) – *Solution*

cont'd

Statement:

$$\forall x(\text{Circle}(x) \to \exists y(\text{Square}(y) \wedge \text{SameColor}(x, y))).$$

Negation:

$$\sim(\forall x(\text{Circle}(x) \to \exists y(\text{Square}(y) \wedge \text{SameColor}(x, y))))$$

$$\equiv \exists x \sim (\text{Circle}(x) \to \exists y(\text{Square}(y) \wedge \text{SameColor}(x, y)))$$

by the law for negating a ∀ statement

$$\equiv \exists x(\text{Circle}(x) \wedge \sim(\exists y(\text{Square}(y) \wedge \text{SameColor}(x, y))))$$

by the law for negating an if-then statement

$$\equiv \exists x(\text{Circle}(x) \wedge \forall y(\sim(\text{Square}(y) \wedge \text{SameColor}(x, y))))$$

by the law for negating a ∃ statement

$$\equiv \exists x(\text{Circle}(x) \wedge \forall y(\sim\text{Square}(y) \vee \sim\text{SameColor}(x, y)))$$

by De Morgan's law

Example 10(d) – *Solution*

cont'd

Statement:

$\exists x(\text{Square}(x) \wedge \forall y(\text{Triangle}(y) \rightarrow \text{RightOf}(x, y)))$.

Negation:

$\sim(\exists x(\text{Square}(x) \wedge \forall y(\text{Triangle}(y) \rightarrow \text{RightOf}(x, y))))$

$\equiv \forall x \sim (\text{Square}(x) \wedge \forall y(\text{Triangle}(x) \rightarrow \text{RightOf}(x, y)))$

by the law for negating a ∃ statement

$\equiv \forall x(\sim\text{Square}(x) \vee \sim(\forall y(\text{Triangle}(y) \rightarrow \text{RightOf}(x, y))))$

by De Morgan's law

$\equiv \forall x(\sim\text{Square}(x) \vee \exists y(\sim(\text{Triangle}(y) \rightarrow \text{RightOf}(x, y))))$

by the law for negating a ∀ statement

$\equiv \forall x(\sim\text{Square}(x) \vee \exists y(\text{Triangle}(y) \wedge \sim\text{RightOf}(x, y)))$

by the law for negating an if-then statement

Formal Logical Notation

The disadvantage of the fully formal notation is that because it is complex and somewhat remote from intuitive understanding, when we use it, we may make errors that go unrecognized.

The advantage, however, is that operations, such as taking negations, can be made completely mechanical and programmed on a computer.

Also, when we become comfortable with formal manipulations, we can use them to check our intuition, and then we can use our intuition to check our formal manipulations.

Formal Logical Notation

Formal logical notation is used in branches of computer science such as artificial intelligence, program verification, and automata theory and formal languages.

Taken together, the symbols for quantifiers, variables, predicates, and logical connectives make up what is known as the **language of first-order logic**.

Even though this language is simpler in many respects than the language we use every day, learning it requires the same kind of practice needed to acquire any foreign language.

Prolog

Prolog

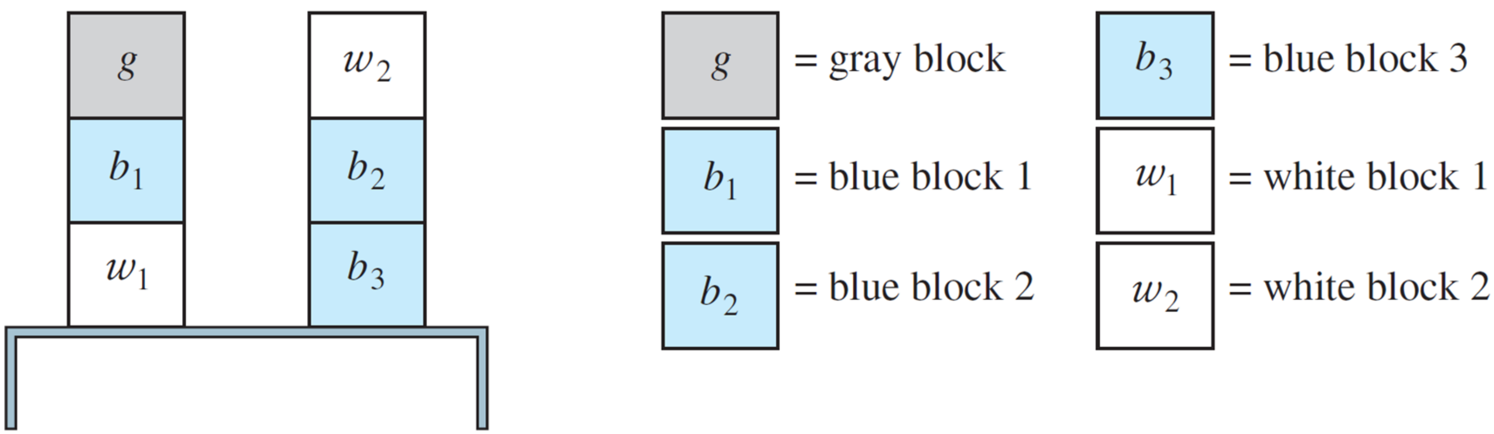
The programming language Prolog (short for *pro*gramming in *log*ic) was developed in France in the 1970s by A. Colmerauer and P. Roussel to help programmers working in the field of artificial intelligence.

A simple Prolog program consists of a set of statements describing some situation together with questions about the situation. Built into the language are search and inference techniques needed to answer the questions by deriving the answers from the given statements.

This frees the programmer from the necessity of having to write separate programs to answer each type of question. Example 11 gives a very simple example of a Prolog program.

Example 11 – *A Prolog Program*

Consider the following picture, which shows colored blocks stacked on a table.



The following are statements in Prolog that describe this picture and ask two questions about it.

isabove(g, b_1) color(g, gray) color(b_3, blue)

Example 11 – *A Prolog Program* cont'd

isabove(b_1, w_1) color(b_1, blue) color(w_1, white)

isabove(w_2, b_2) color(b_2, blue) color(w_2, white)

isabove(b_2, b_3) isabove(X, Z) if isabove(X, Y) and isabove(Y, Z)

?color(b_1, blue) ?isabove(X, w_1)

The statements "isabove(g, b_1)" and "color(g, gray)" are to be interpreted as "g is above b_1" and "g is colored gray". The statement "isabove(X, Z) if isabove(X, Y) and isabove(Y, Z)" is to be interpreted as "For all X, Y, and Z, if X is above Y and Y is above Z, then X is above Z."

Example 11 – *A Prolog Program* cont'd

The program statement

$$?\text{color}(b_1, \text{blue})$$

is a question asking whether block b_1 is colored blue. Prolog answers this by writing

Yes.

The statement

$$?\text{isabove}(X, w_1)$$

is a question asking for which blocks X the predicate "X is above w_1" is true.

Example 11 – *A Prolog Program* cont'd

Prolog answers by giving a list of all such blocks. In this case, the answer is

$$X = b_1,\ X = g.$$

Note that Prolog can find the solution $X = b_1$ by merely searching the original set of given facts. However, Prolog must *infer* the solution $X = g$ from the following statements:

isabove(g, b_1),

isabove(b_1,w_1),

isabove(X, Z) if isabove(X, Y) and isabove(Y, Z).

Example 11 – *A Prolog Program* cont'd

Write the answers Prolog would give if the following questions were added to the program above.

a. ?isabove(b_2, w_1) **b**. ?color(w_1, X) **c**. ?color(X, blue)

Solution:

a. The question means "Is b_2 above w_1?"; so the answer is "No."

b. The question means "For what colors X is the predicate 'w_1 is colored X' true?"; so the answer is "X = white."

Example 11 – *Solution*

cont'd

c. The question means "For what blocks is the predicate 'X is colored blue' true?"; so the answer is "$X = b_1$," "$X = b_2$," and "$X = b_3$."