CHAPTER 3

THE LOGIC OF QUANTIFIED STATEMENTS

SECTION 3.2

Predicates and Quantified Statements II

Negations of Quantified Statements

Negations of Quantified Statements

The general form of the negation of a universal statement follows immediately from the definitions of negation and of the truth values for universal and existential statements.

Theorem 3.2.1 Negation of a Universal Statement

The negation of a statement of the form

$$\forall x \text{ in } D, Q(x)$$

is logically equivalent to a statement of the form

$$\exists x \text{ in } D \text{ such that } \sim Q(x).$$

Symbolically, $\sim(\forall x \in D, Q(x)) \equiv \exists x \in D \text{ such that } \sim Q(x).$

Negations of Quantified Statements

Thus

The negation of a universal statement ("all are") is logically equivalent to an existential statement ("some are not" or "there is at least one that is not").

Note that when we speak of **logical equivalence for quantified statements,** we mean that the statements always have identical truth values no matter what predicates are substituted for the predicate symbols and no matter what sets are used for the domains of the predicate variables.

Negations of Quantified Statements

The general form for the negation of an existential statement follows immediately from the definitions of negation and of the truth values for existential and universal statements.

Theorem 3.2.2 Negation of an Existential Statement

The negation of a statement of the form

$$\exists x \text{ in } D \text{ such that } Q(x)$$

is logically equivalent to a statement of the form

$$\forall x \text{ in } D, \sim Q(x).$$

Symbolically, $\sim(\exists x \in D \text{ such that } Q(x)) \equiv \forall x \in D, \sim Q(x).$

Negations of Quantified Statements

Thus

The negation of an existential statement ("some are") is logically equivalent to a universal statement ("none are" or "all are not").

Example 1 – *Negating Quantified Statements*

Write formal negations for the following statements:

a. ∀ primes *p*, *p* is odd.

b. ∃ a triangle *T* such that the sum of the angles of *T* equals 200°.

Solution:

a. By applying the rule for the negation of a ∀ statement, you can see that the answer is

∃a prime *p* such that *p* is not odd.

Example 1 – *Solution*

cont'd

b. By applying the rule for the negation of a ∃ statement, you can see that the answer is

∀ triangles *T*, the sum of the angles of *T* does not equal 200°.

Negations of Universal Conditional Statements

Negations of Universal Conditional Statements

Negations of universal conditional statements are of special importance in mathematics.

The form of such negations can be derived from facts that have already been established.

By definition of the negation of a *for all* statement,

$$\sim(\forall x, P(x) \rightarrow Q(x)) \equiv \exists x \text{ such that } \sim(P(x) \rightarrow Q(x)). \qquad 3.2.1$$

But the negation of an if-then statement is logically equivalent to an *and* statement. More precisely,

$$\sim(P(x) \rightarrow Q(x)) \equiv P(x) \wedge \sim Q(x). \qquad 3.2.2$$

Negations of Universal Conditional Statements

Substituting (3.2.2) into (3.2.1) gives

$$\sim(\forall x,\ P(x) \rightarrow Q(x)) \equiv \exists x \text{ such that } (P(x) \wedge \sim Q(x)).$$

Written less symbolically, this becomes

Negation of a Universal Conditional Statement

$$\sim(\forall x, \text{ if } P(x) \text{ then } Q(x)) \equiv \exists x \text{ such that } P(x) \text{ and } \sim Q(x).$$

Example 4 – *Negating Universal Conditional Statements*

Write a formal negation for statement (a) and an informal negation for statement (b).

a. ∀ people *p*, if *p* is blond then *p* has blue eyes.

b. If a computer program has more than 100,000 lines, then it contains a bug.

Solution:

a. ∃ a person *p* such that *p* is blond and *p* does not have blue eyes.

Example 4 – *Solution*

cont'd

b. There is at least one computer program that has more than 100,000 lines and does not contain a bug.

The Relation among ∀, ∃, ∧, and ∨

The Relation among ∀, ∃, ∧, and ∨

The negation of a *for all* statement is a *there exists* statement, and the negation of a *there exists* statement is a *for all* statement.

These facts are analogous to De Morgan's laws, which state that the negation of an *and* statement is an *or* statement and that the negation of an *or* statement is an *and* statement.

This similarity is not accidental. In a sense, universal statements are generalizations of *and* statements, and existential statements are generalizations of *or* statements.

The Relation among ∀, ∃, ∧, and ∨

If $Q(x)$ is a predicate and the domain D of x is the set $\{x_1, x_2, \ldots, x_n\}$, then the statements

$$\forall x \in D, Q(x)$$

and

$$Q(x_1) \wedge Q(x_2) \wedge \cdots \wedge Q(x_n)$$

are logically equivalent.

The Relation among ∀, ∃, ∧, and ∨

Similarly, if $Q(x)$ is a predicate and $D = \{x_1, x_2, \ldots, x_n\}$, then the statements

$$\exists x \in D \text{ such that } Q(x)$$

and

$$Q(x_1) \vee Q(x_2) \vee \cdots \vee Q(x_n)$$

are logically equivalent.

Vacuous Truth of Universal Statements

Vacuous Truth of Universal Statements

Suppose a bowl sits on a table and next to the bowl is a pile of five blue and five gray balls, any of which may be placed in the bowl.

If three blue balls and one gray ball are placed in the bowl, as shown in Figure 3.2.1(a), the statement “All the balls in the bowl are blue” would be false (since one of the balls in the bowl is gray).

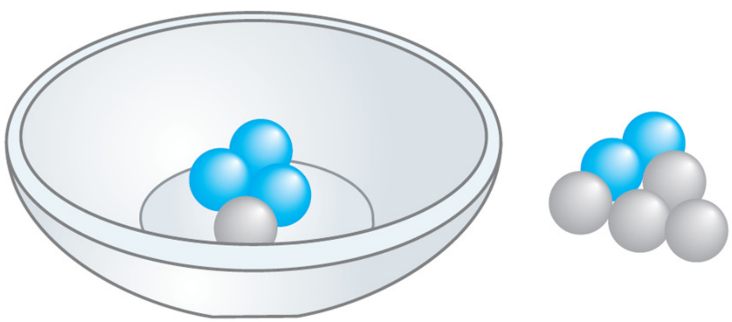

Figure 3.2.1(a)

Vacuous Truth of Universal Statements

Now suppose that no balls at all are placed in the bowl, as shown in Figure 3.2.1(b).

Consider the statement

All the balls in the bowl are blue.

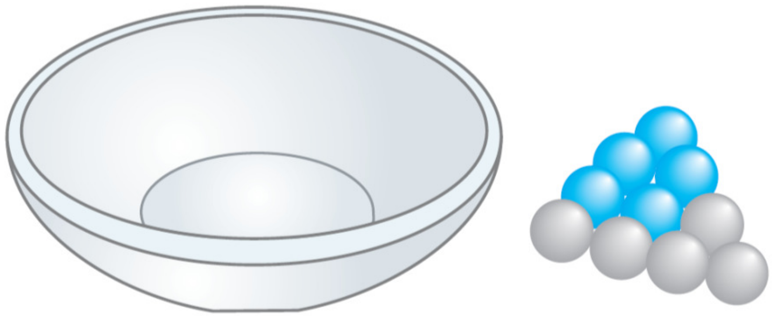

Figure 3.2.1(b)

Vacuous Truth of Universal Statements

Is this statement true or false? The statement is false if, and only if, its negation is true.

And its negation is

There exists a ball in the bowl that is not blue.

But the only way this negation can be true is for there actually to be a nonblue ball in the bowl.

And there is not! Hence the negation is false, and so the statement is true "by default."

Vacuous Truth of Universal Statements

In general, a statement of the form

$$\forall x \text{ in } D, \text{ if } P(x) \text{ then } Q(x)$$

is called **vacuously true** or **true by default** if, and only if, $P(x)$ is false for every x in D.

Variants of Universal Conditional Statements

Variants of Universal Conditional Statements

We have known that a conditional statement has a contrapositive, a converse, and an inverse.

The definitions of these terms can be extended to universal conditional statements.

• Definition

Consider a statement of the form: $\forall x \in D$, if $P(x)$ then $Q(x)$.

1. Its **contrapositive** is the statement: $\forall x \in D$, if $\sim Q(x)$ then $\sim P(x)$.
2. Its **converse** is the statement: $\forall x \in D$, if $Q(x)$ then $P(x)$.
3. Its **inverse** is the statement: $\forall x \in D$, if $\sim P(x)$ then $\sim Q(x)$.

Example 5 – *Contrapositive, Converse, and Inverse of a Universal Conditional Statement*

Write a formal and an informal contrapositive, converse, and inverse for the following statement:

If a real number is greater than 2, then its square is greater than 4.

Solution:

The formal version of this statement is

$$\forall x \in \mathbf{R}, \text{ if } x > 2 \text{ then } x^2 > 4.$$

Example 5 – *Solution*

cont'd

Contrapositive: $\forall x \in \mathbf{R}$, if $x^2 \leq 4$ then $x \leq 2$.

Or: If the square of a real number is less than or equal to 4, then the number is less than or equal to 2.

Converse: $\forall x \in \mathbf{R}$, if $x^2 > 4$ then $x > 2$.

Or: If the square of a real number is greater than 4, then the number is greater than 2.

Inverse: $\forall x \in \mathbf{R}$, if $x \leq 2$ then $x^2 \leq 4$.

Or: If a real number is less than or equal to 2, then the square of the number is less than or equal to 4.

Variants of Universal Conditional Statements

Let $P(x)$ and $Q(x)$ be any predicates, let D be the domain of x, and consider the statement

$$\forall x \in D, \text{if } P(x) \text{ then } Q(x)$$

and its contrapositive

$$\forall x \in D, \text{if } \sim Q(x) \text{ then } \sim P(x).$$

Any particular x in D that makes "if $P(x)$ then $Q(x)$" true also makes "if $\sim Q(x)$ then $\sim P(x)$" true (by the logical equivalence between $p \to q$ and $\sim q \to \sim p$).

Variants of Universal Conditional Statements

It follows that the sentence "If $P(x)$ then $Q(x)$" is true for all x in D if, and only if, the sentence "If $\sim Q(x)$ then $\sim P(x)$" is true for all x in D.

Thus we write the following and say that a universal conditional statement is logically equivalent to its contrapositive:

$$\forall x \in D, \text{ if } P(x) \text{ then } Q(x) \equiv \forall x \in D, \text{ if } \sim Q(x) \text{ then } \sim P(x)$$

Variants of Universal Conditional Statements

In Example 3.2.5 we noted that the statement

$\forall x \in \mathbf{R}$, if $x > 2$ then $x^2 > 4$

has the converse

$\forall x \in \mathbf{R}$, if $x^2 > 4$ then $x > 2$.

Observe that the statement is true whereas its converse is false (since, for instance, $(-3)^2 = 9 > 4$ but $-3 \not> 2$).

Variants of Universal Conditional Statements

This shows that a universal conditional statement may have a different truth value from its converse.

Hence a universal conditional statement is not logically equivalent to its converse.

This is written in symbols as follows:

$$\forall x \in D, \text{ if } P(x) \text{ then } Q(x) \;\not\equiv\; \forall x \in D, \text{ if } Q(x) \text{ then } P(x).$$

Necessary and Sufficient Conditions, Only If

Necessary and Sufficient Conditions, Only If

The definitions of *necessary, sufficient,* and *only if* can also be extended to apply to universal conditional statements.

• Definition

- "$\forall x$, $r(x)$ is a **sufficient condition** for $s(x)$" means "$\forall x$, if $r(x)$ then $s(x)$."
- "$\forall x$, $r(x)$ is a **necessary condition** for $s(x)$" means "$\forall x$, if $\sim r(x)$ then $\sim s(x)$" or, equivalently, "$\forall x$, if $s(x)$ then $r(x)$."
- "$\forall x$, $r(x)$ **only if** $s(x)$" means "$\forall x$, if $\sim s(x)$ then $\sim r(x)$" or, equivalently, "$\forall x$, if $r(x)$ then $s(x)$."

Example 6 – *Necessary and Sufficient Conditions*

Rewrite the following statements as quantified conditional statements. Do not use the word *necessary* or *sufficient*.

a. Squareness is a sufficient condition for rectangularity.

b. Being at least 35 years old is a necessary condition for being President of the United States.

Solution:

a. A formal version of the statement is

$$\forall x, \text{ if } x \text{ is a square, then } x \text{ is a rectangle.}$$

Example 6 – *Solution*

cont'd

Or, in informal language:

If a figure is a square, then it is a rectangle.

b. Using formal language, you could write the answer as

$\forall$ people *x*, if *x* is younger than 35, then *x* cannot be President of the United States.

Or, by the equivalence between a statement and its contrapositive:

$\forall$ people *x*, if *x* is President of the United States, then *x* is at least 35 years old.