CHAPTER 2

THE LOGIC OF COMPOUND STATEMENTS

SECTION 2.5

Application: Number Systems and Circuits for Addition

Application: Number Systems and Circuits for Addition

In elementary school, you learned the meaning of decimal notation: that to interpret a string of decimal digits as a number, you mentally multiply each digit by its place value.

For instance, 5,049 has a 5 in the thousands place, a 0 in the hundreds place, a 4 in the tens place, and a 9 in the ones place. Thus

$$5{,}049 = 5 \cdot (1{,}000) + 0 \cdot (100) + 4 \cdot (10) + 9 \cdot (1).$$

Application: Number Systems and Circuits for Addition

Using exponential notation, this equation can be rewritten as

$$5{,}049 = 5 \cdot 10^3 + 0 \cdot 10^2 + 4 \cdot 10^1 + 9 \cdot 10^0.$$

More generally, decimal notation is based on the fact that any positive integer can be written uniquely as a sum of products of the form

$$d \cdot 10^n,$$

where each n is a nonnegative integer and each d is one of the decimal digits 0, 1, 2, 3, 4, 5, 6, 7, 8, or 9.

Application: Number Systems and Circuits for Addition

The word *decimal* comes from the Latin root *deci*, meaning “ten.” Decimal (or base 10) notation expresses a number as a string of digits in which each digit’s position indicates the power of 10 by which it is multiplied.

The right-most position is the ones place (or 10^0 place), to the left of that is the tens place (or 10^1 place), to the left of that is the hundreds place (or 10^2 place), and so forth, as illustrated below.

Place	10^3 thousands	10^2 hundreds	10^1 tens	10^0 ones
Decimal Digit	5	0	4	9

Binary Representation of Numbers

Binary Representation of Numbers

In computer science, **base 2 notation,** or **binary notation,** is of special importance because the signals used in modern electronics are always in one of only two states. (The Latin root *bi* means “two.”)

We can show that any integer can be represented uniquely as a sum of products of the form

$$d \cdot 2^n,$$

where each n is an integer and each d is one of the binary digits (or bits) 0 or 1.

Binary Representation of Numbers

For example,

$$27 = 16 + 8 + 2 + 1$$

$$= 1 \cdot 2^4 + 1 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0.$$

The places in binary notation correspond to the various powers of 2.

Binary Representation of Numbers

The right-most position is the ones place (or 2^0 place), to the left of that is the twos place (or 2^1 place), to the left of that is the fours place (or 2^2 place), and so forth, as illustrated below.

Place	2^4 sixteens	2^3 eights	2^2 fours	2^1 twos	2^0 ones
Binary Digit	1	1	0	1	1

As in the decimal notation, leading zeros may be added or dropped as desired. For example,

$$003_{10} = 3_{10} = 1 \cdot 2^1 + 1 \cdot 2^0 = 11_2 = 011_2.$$

Binary Representation of Numbers

A list of powers of 2 is useful for doing binary-to-decimal and decimal-to-binary conversions. See Table 2.5.1.

Power of 2	2^{10}	2^9	2^8	2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0
Decimal Form	1024	512	256	128	64	32	16	8	4	2	1

Powers of 2

Table 2.5.1

Example 2 – *Converting a Binary to a Decimal Number*

Represent 110101_2 in decimal notation.

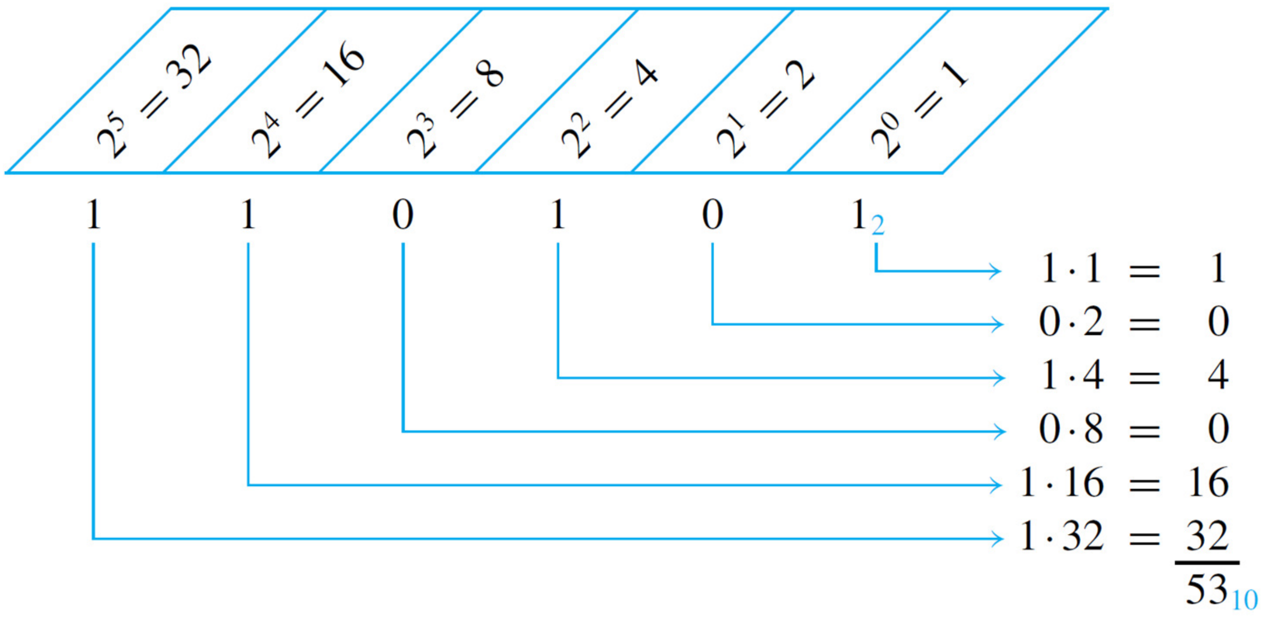
Solution:

$$\begin{aligned} 110101_2 &= 1 \cdot 2^5 + 1 \cdot 2^4 + 0 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0 \\ &= 32 + 16 + 4 + 1 \\ &= 53_{10} \end{aligned}$$

Example 2 – *Solution*

cont'd

Alternatively, the schema below may be used.



Example 3 – *Converting a Decimal to a Binary Number*

Represent 209 in binary notation.

Solution:

Use Table 2.5.1 to write 209 as a sum of powers of 2, starting with the highest power of 2 that is less than 209 and continuing to lower powers.

Power of 2	2^{10}	2^9	2^8	2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0
Decimal Form	1024	512	256	128	64	32	16	8	4	2	1

Powers of 2

Table 2.5.1

Example 3 – *Solution*

cont'd

Since 209 is between 128 and 256, the highest power of 2 that is less than 209 is 128. Hence

$$209_{10} = 128 + \text{a smaller number.}$$

Now 209 – 128 = 81, and 81 is between 64 and 128, so the highest power of 2 that is less than 81 is 64. Hence

$$209_{10} = 128 + 64 + \text{a smaller number.}$$

Example 3 – *Solution*

cont'd

Continuing in this way, you obtain

$$\begin{aligned}209_{10} &= 128 + 64 + 16 + 1 \\ &= 1\cdot 2^7 + 1\cdot 2^6 + 0\cdot 2^5 + 1\cdot 2^4 + 0\cdot 2^3 + 0\cdot 2^2 + 0\cdot 2^1 + 1\cdot 2^0.\end{aligned}$$

For each power of 2 that occurs in the sum, there is a 1 in the corresponding position of the binary number.

Example 3 – *Solution*

cont'd

For each power of 2 that is missing from the sum, there is a 0 in the corresponding position of the binary number.

Thus

$$209_{10} = 11010001_2$$

Binary Addition and Subtraction

Example 4 – *Addition in Binary Notation*

Add 1101_2 and 111_2 using binary notation.

Solution:

Because $2_{10} = 10_2$ and $1_{10} = 1_2$, the translation of $1_{10} + 1_{10} = 2_{10}$ to binary notation is

$$\begin{array}{r} 1_2 \\ +\quad 1_2 \\ \hline 10_2 \end{array}$$

It follows that adding two 1's together results in a carry of 1 when binary notation is used.

Example 4 – *Solution*

cont'd

Adding three 1's together also results in a carry of 1 since $3_{10} = 11_2$ ("one one base two").

$$\begin{array}{rr} & 1_2 \\ + & 1_2 \\ + & 1_2 \\ \hline & 11_2 \end{array}$$

Thus the addition can be performed as follows:

$$\begin{array}{rccccl} & & 1 & 1 & 1 & & \leftarrow \text{carry row} \\ & & 1 & 1 & 0 & 1_2 & \\ + & & & 1 & 1 & 1_2 & \\ \hline & 1 & 0 & 1 & 0 & 0_2 & \end{array}$$

Example 5 – *Subtraction in Binary Notation*

Subtract 1011_2 from 11000_2 using binary notation.

Solution:

In decimal subtraction the fact that $10_{10} - 1_{10} = 9_{10}$ is used to borrow across several columns. For example, consider the following:

$$\begin{array}{rccccl}
 & & 9 & 9 & & \\
 & & \cancel{1} & & 1 & \leftarrow \text{borrow row} \\
 & \cancel{1} & \cancel{0} & \cancel{0} & 0_{10} & \\
- & & & 5 & 8_{10} & \\ \hline
 & & 9 & 4 & 2_{10} &
\end{array}$$

Example 5 – *Solution*

cont'd

In binary subtraction it may also be necessary to borrow across more than one column. But when you borrow a 1_2 from 10_2, what remains is 1_2.

$$\begin{array}{rr} & 10_2 \\ - & 1_2 \\ \hline & 1_2 \end{array}$$

Thus the subtraction can be performed as follows:

$$\begin{array}{rccccc l} & & 0 & 1 & 1 & & \\ & & & 1 & 1 & 1 & \leftarrow \text{borrow row} \\ & 1 & \cancel{1} & \cancel{0} & \cancel{0} & 0_2 & \\ - & & 1 & 0 & 1 & 1_2 & \\ \hline & & 1 & 1 & 0 & 1_2 & \end{array}$$

Circuits for Computer Addition

Circuits for Computer Addition

Consider the question of designing a circuit to produce the sum of two binary digits *P* and *Q*. Both *P* and *Q* can be either 0 or 1. And the following facts are known:

$$\begin{aligned} 1_2 + 1_2 & \qquad\quad = 10_2, \\ 1_2 + 0_2 &= 1_2 = 01_2, \\ 0_2 + 1_2 &= 1_2 = 01_2, \\ 0_2 + 0_2 &= 0_2 = 00_2. \end{aligned}$$

Circuits for Computer Addition

It follows that the circuit to be designed must have two outputs—one for the left binary digit (this is called the **carry**) and one for the right binary digit (this is called the **sum**).

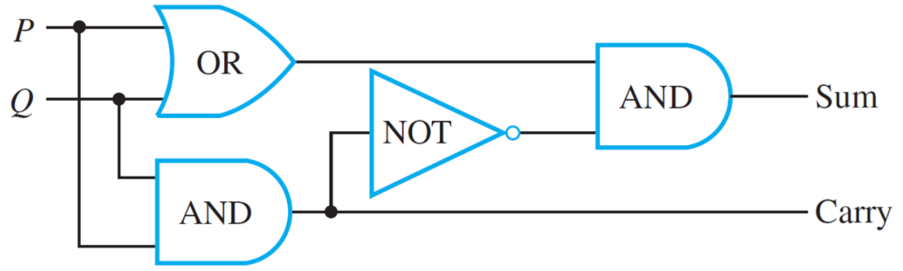
The carry output is 1 if both P and Q are 1; it is 0 otherwise. Thus the carry can be produced using the AND-gate circuit that corresponds to the Boolean expression $P \wedge Q$. The sum output is 1 if either P or Q, but not both, is 1.

Circuits for Computer Addition

The sum can, therefore, be produced using a circuit that corresponds to the Boolean expression for *exclusive or*: $(P \vee Q) \wedge \sim (P \wedge Q)$. Hence, a circuit to add two binary digits P and Q can be constructed as in Figure 2.5.1. This circuit is called a **half-adder.**

HALF-ADDER

Circuit



Input/Output Table

P	Q	Carry	Sum
1	1	1	0
1	0	0	1
0	1	0	1
0	0	0	0

Circuit to Add $P + Q$, Where P and Q Are Binary Digits
Figure 2.5.1

Circuits for Computer Addition

In order to construct a circuit that will add multidigit binary numbers, it is necessary to incorporate a circuit that will compute the sum of three binary digits. Such a circuit is called a **full-adder.**

Consider a general addition of three binary digits *P*, *Q*, and *R* that results in a carry (or left-most digit) *C* and a sum (or right-most digit) *S*.

$$\begin{array}{rr} & P \\ + & Q \\ + & R \\ \hline & CS \end{array}$$

Circuits for Computer Addition

The operation of the full-adder is based on the fact that addition is a binary operation: Only two numbers can be added at one time. Thus *P* is first added to *Q* and then the result is added to *R*. For instance, consider the following addition:

$$\left.\begin{array}{r} \left.\begin{array}{r} 1_2 \\ +\ 0_2 \end{array}\right\} 1_2 + 0_2 = 01_2 \\ +\ 1_2 \qquad\qquad\qquad\quad \end{array}\right\} 1_2 + 1_2 = 10_2$$

$$\begin{array}{r} \hline 10_2 \end{array}$$

Circuits for Computer Addition

The process illustrated here can be broken down into steps that use half-adder circuits.

Step 1: Add *P* and *Q* using a half-adder to obtain a binary number with two digits.

$$\begin{array}{r} P \\ +\quad Q \\ \hline C_1 S_1 \end{array}$$

Circuits for Computer Addition

Step 2: Add *R* to the sum C_1S_1 of *P* and *Q*.

$$\begin{array}{r} C_1S_1 \\ +\quad R \\ \hline \end{array}$$

To do this, proceed as follows:

Step 2a: Add *R* to S_1 using a half-adder to obtain the two-digit number C_2S.

$$\begin{array}{r} S_1 \\ +\quad R \\ \hline C_2S \end{array}$$

Then *S* is the right-most digit of the entire sum of *P*, *Q*, and *R*.

Circuits for Computer Addition

Step 2b: Determine the left-most digit, C, of the entire sum as follows: First note that it is impossible for both C_1 and C_2 to be 1's. For if $C_1 = 1$, then P and Q are both 1, and so $S_1 = 0$. Consequently, the addition of S_1 and R gives a binary number C_2S_1 where $C_2 = 0$.

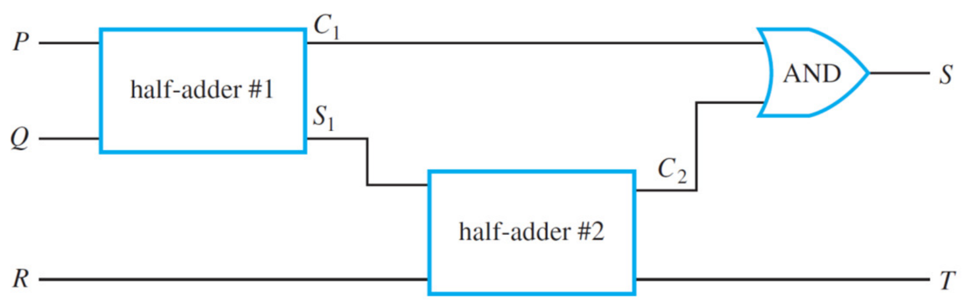
Next observe that C will be a 1 in the case that the addition of P and Q gives a carry of 1 or in the case that the addition of S_1 (the right-most digit of $P + Q$) and R gives a carry of 1.

Circuits for Computer Addition

In other words, $C = 1$ if, and only if, $C_1 = 1$ or $C_2 = 1$. It follows that the circuit shown in Figure 2.5.2 will compute the sum of three binary digits.

FULL-ADDER

Circuit



Input/Output Table

P	Q	R	C	S
1	1	1	1	1
1	1	0	1	0
1	0	1	1	0
1	0	0	0	1
0	1	1	1	0
0	1	0	0	1
0	0	1	0	1
0	0	0	0	0

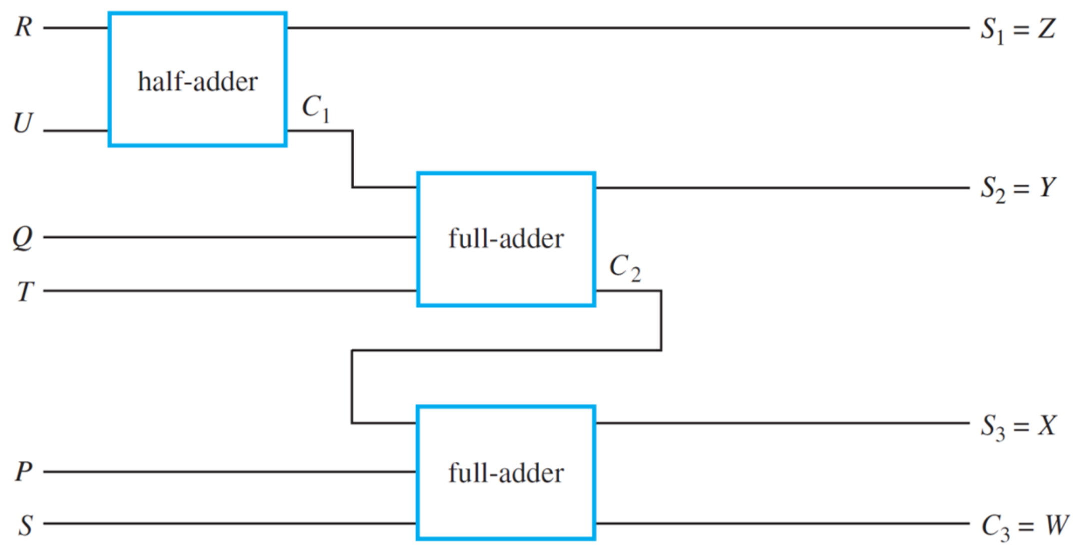
Circuit to Add $P + Q + R$, Where P, Q, and R Are Binary Digits

Figure 2.5.2

Circuits for Computer Addition

Two full-adders and one half-adder can be used together to build a circuit that will add two three-digit binary numbers *PQR* and *STU* to obtain the sum *WXYZ*. This is illustrated in Figure 2.5.3. Such a circuit is called a **parallel adder.**

Parallel adders can be constructed to add binary numbers of any finite length.



A Parallel Adder to Add *PQR* and *STU* to Obtain *WXYZ*

Figure 2.5.3

Two’s Complements and the Computer Representation of Negative Integers

Two’s Complements and the Computer Representation of Negative Integers

Typically, a fixed number of bits is used to represent integers on a computer, and these are required to represent negative as well as nonnegative integers.

Sometimes a particular bit, normally the left-most, is used as a sign indicator, and the remaining bits are taken to be the absolute value of the number in binary notation.

The problem with this approach is that the procedures for adding the resulting numbers are somewhat complicated and the representation of 0 is not unique.

Two’s Complements and the Computer Representation of Negative Integers

A more common approach, using *two’s complements*, makes it possible to add integers quite easily and results in a unique representation for 0. The two’s complement of an integer relative to a fixed bit length is defined as follows:

• Definition

Given a positive integer a, the **two’s complement of *a* relative to a fixed bit length** n is the n-bit binary representation of

$$2^n - a.$$

Two’s Complements and the Computer Representation of Negative Integers

There is a convenient way to compute two’s complements that involves less arithmetic than direct application of the definition. For an 8-bit representation, it is based on three facts:

1. $2^8 - a = [(2^8 - 1) - a] + 1.$

2. The binary representation of $2^8 - 1$ is 11111111_2.

3. Subtracting an 8-bit binary number *a* from 11111111_2 just switches all the 0’s in *a* to 1’s and all the 1’s to 0’s. (The resulting number is called the **one’s complement** of the given number.)

Two’s Complements and the Computer Representation of Negative Integers

In general,

To find the 8-bit two’s complement of a positive integer a that is at most 255:

- Write the 8-bit binary representation for a.
- Flip the bits (that is, switch all the 1’s to 0’s and all the 0’s to 1’s).
- Add 1 in binary notation.

Example 6 – *Finding a Two's Complement*

Find the 8-bit two's complement of 19.

Solution:

Write the 8-bit binary representation for 19, switch all the 0's to 1's and all the 1's to 0's, and add 1.

$$19_{10} = (16 + 2 + 1)_{10}$$

$$= 00010011_2 \xrightarrow{\text{flip the bits}} 11101100 \xrightarrow{\text{add } 1} 11101101$$

Example 6 – *Solution*

cont'd

To check this result, note that

$$11101101_2 = (128 + 64 + 32 + 8 + 4 + 1)_{10}$$

$$= 237_{10}$$

$$= (256 - 19)_{10}$$

$$= (2^8 - 19)_{10},$$

which *is* the two's complement of 19.

Two’s Complements and the Computer Representation of Negative Integers

Observe that because

$$2^8 - (2^8 - a) = a$$

the *two’s complement of the two’s complement of a number is the number itself*, and therefore,

To find the decimal representation of the integer with a given 8-bit two’s complement:

- Find the two’s complement of the given two’s complement.
- Write the decimal equivalent of the result.

Example 7 – *Finding a Number with a Given Two's Complement*

What is the decimal representation for the integer with two's complement 10101001?

Solution:

$$10101001_2 \xrightarrow{\text{flip the bits}} 01010110$$

$$\xrightarrow{\text{add 1}} 01010111_2 = (64 + 16 + 4 + 2 + 1)_{10}$$

$$= 87_{10}$$

Example 7 – *Solution*

cont'd

To check this result, note that the given number is

$$\begin{aligned} 10101001_2 &= (128 + 32 + 8 + 1)_{10} \\ &= 169_{10} \\ &= (256 - 87)_{10} \\ &= (2^8 - 87)_{10}, \end{aligned}$$

which is the two's complement of 87.

8-Bit Representation of a Number

8-Bit Representation of a Number

Now consider the two's complement of an integer n that satisfies the inequality $1 \leq n \leq 128$. Then

$$-1 \geq -n \geq -128$$

because multiplying by -1 reverses the direction of the inequality

and

$$2^8 - 1 \geq 2^8 - n \geq 2^8 - 128$$

by adding 2^8 to all parts of the inequality.

But $2^8 - 128 = 256 - 128 = 128 = 2^7$. Hence

$$2^7 \leq \text{the two's complement of } n < 2^8.$$

8-Bit Representation of a Number

It follows that the 8-bit two’s complement of an integer from 1 through 128 has a leading bit of 1. Note also that the ordinary 8-bit representation of an integer from 0 through 127 has a leading bit of 0.

Consequently, eight bits can be used to represent both nonnegative and negative integers by representing each nonnegative integer up through 127 using ordinary 8-bit binary notation and representing each negative integer from –1 through –128 as the two’s complement of its absolute value.

8-Bit Representation of a Number

That is, for any integer *a* from –128 through 127,

The 8-bit representation of a

$$= \begin{cases} \text{the 8-bit binary representation of } a & \text{if } a \geq 0 \\ \text{the 8-bit binary representation of } 2^8 - |a| & \text{if } a < 0 \end{cases}.$$

8-Bit Representation of a Number

The representations are illustrated in Table 2.5.2.

Integer	8-Bit Representation (ordinary 8-bit binary notation if nonnegative or 8-bit two's complement of absolute value if negative)	Decimal Form of Two's Complement for Negative Integers
127	01111111	
126	01111110	
⋮	⋮	
2	00000010	
1	00000001	
0	00000000	
-1	11111111	$2^8 - 1$
-2	11111110	$2^8 - 2$
-3	11111101	$2^8 - 3$
⋮	⋮	⋮
-127	10000001	$2^8 - 127$
-128	10000000	$2^8 - 128$

Table 2.5.2

Computer Addition with Negative Integers

Computer Addition with Negative Integers

To add two integers in the range -128 through 127 whose sum is also in the range -128 through 127:

- Convert both integers to their 8-bit representations (representing negative integers by using the two's complements of their absolute values).
- Add the resulting integers using ordinary binary addition.
- Truncate any leading 1 (overflow) that occurs in the 2^8th position.
- Convert the result back to decimal form (interpreting 8-bit integers with leading 0's as nonnegative and 8-bit integers with leading 1's as negative).

Computer Addition with Negative Integers

Case 1, (both integers are nonnegative): This case is easy because if two nonnegative integers from 0 through 127 are written in their 8-bit representations and if their sum is also in the range 0 through 127, then the 8-bit representation of their sum has a leading 0 and is therefore interpreted correctly as a nonnegative integer.

The example below illustrates what happens when 38 and 69 are added.

$$\begin{array}{llr}
 & \boxed{0\,|\,0\,|\,1\,|\,0\,|\,0\,|\,1\,|\,1\,|\,0} & 38 \\
+ & & \\
 & \boxed{0\,|\,1\,|\,0\,|\,0\,|\,0\,|\,1\,|\,0\,|\,1} & 69 \\
\hline
 & \boxed{0\,|\,1\,|\,1\,|\,0\,|\,1\,|\,0\,|\,1\,|\,1} & 107
\end{array}$$

Computer Addition with Negative Integers

To be concrete, let the nonnegative integer be *a* and the negative integer be $-b$ and suppose both *a* and $-b$ are in the range -128 through 127. The crucial observation is that adding the 8-bit representations of *a* and $-b$ is equivalent to computing

$$a + (2^8 - b)$$

because the 8-bit representation of $-b$ is the binary representation of $2^8 - b$.

Computer Addition with Negative Integers

Case 2* (*a is nonnegative and* $-$*b is negative and* |*a*| < |*b*|)*: In this case, observe that a = |a| < /b/ = b and

$$a + (2^8 - b) = 2^8 - (b - a),$$

and the binary representation of this number is the 8-bit representation of $-(b - a) = a + (-b)$. We must be careful to check that $2^8 - (b - a)$ is between 2^7 and 2^8. But it *is* because

$$2^7 = 2^8 - 2^7 \leq 2^8 - (b - a) < 2^8 \qquad \text{since } 0 < b - a \leq \text{b} \leq 128 = 2^7.$$

Hence in case $|a| < |b|$, adding the 8-bit representations of a and $-b$ gives the 8-bit representation of $a + (-b)$.

Example 8 – *Computing $a + (-b)$ Where $0 \leq a < b \leq 128$*

Use 8-bit representations to compute 39 + (–89).

Solution:

Step 1: Change from decimal to 8-bit representations using the two's complement to represent –89.

Since $39_{10} = (32 + 4 + 2 + 1)_{10} = 100111_2$, the 8-bit representation of 39 is 00100111.

Now the 8-bit representation of –89 is the two's complement of 89.

Example 8 – *Solution*

cont'd

This is obtained as follows:

$$89_{10} = (64 + 16 + 8 + 1)_{10} = 01011001_2 \xrightarrow{\text{flip the bits}} 10100110 \xrightarrow{\text{add } 1} 10100111$$

So the 8-bit representation of –89 is 10100111.

Example 8 – *Solution*

cont'd

Step 2: Add the 8-bit representations in binary notation and truncate the 1 in the 2^8th position if there is one:

$$
\begin{array}{r|c|c|c|c|c|c|c|c|}
\cline{2-9}
 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\
\cline{2-9}
\multicolumn{1}{r}{+} \\
\cline{2-9}
 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\
\cline{2-9}
\hline
\cline{2-9}
\text{There is no 1 in the } 2^8\text{th position to truncate} \rightarrow & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\
\cline{2-9}
\end{array}
$$

Example 8 – *Solution*

cont'd

Step 3: Find the decimal equivalent of the result. Since its leading bit is 1, this number is the 8-bit representation of a negative integer.

$$11001110 \xrightarrow{\text{flip the bits}} 00110001 \xrightarrow{\text{add } 1} 00110010$$
$$\leftrightarrow -(32+16+2)_{10} = -50_{10}$$

Note that since 39 – 89 = –50, this procedure gives the correct answer.

Computer Addition with Negative Integers

***Case 3* (*a is nonnegative and –b is negative and |b| ≤ |a|*):** In this case, observe that b = |b| ≤ /a/ = a and

$$a + (2^8 - b) \;=\; 2^8 + (a - b).$$

Also

$$2^8 \;\leq\; 2^8 + (a - b) \;<\; 2^8 + 2^7 \qquad \text{because } 0 \leq a - b \leq \text{a} < 128 = 2^7.$$

So the binary representation of $a + (2^8 - b) \;=\; 2^8 + (a - b)$ has a leading 1 in the ninth (2^8th) position. This leading 1 is often called "overflow" because it does not fit in the 8-bit integer format.

Computer Addition with Negative Integers

Now subtracting 2^8 from $2^8 + (a - b)$ is equivalent to truncating the leading 1 in the 2^8th position of the binary representation of the number. But

$$[a + (2^8 - b)] - 2^8 \;=\; 2^8 + (a - b) - 2^8 \;=\; a - b \;=\; a + (-b).$$

Hence in case $|a| \geq |b|$, adding the 8-bit representations of a and $-b$ and truncating the leading 1 (which is sure to be present) gives the 8-bit representation of $a + (-b)$.

Example 9 – *Computing a + (–b) Where 1 ≤ b < a ≤ 127*

Use 8-bit representations to compute 39 + (–25).

Solution:

Step 1: Change from decimal to 8-bit representations using the two's complement to represent –25.

As in Example 8, the 8-bit representation of 39 is 00100111. Now the 8-bit representation of –25 is the two's complement of 25, which is obtained as follows:

$$25_{10} = (16 + 8 + 1)_{10} = 00011001_2 \xrightarrow{\text{flip the bits}} 11100110 \xrightarrow{\text{add } 1} 11100111$$

Example 9 – *Solution*

cont’d

So the 8-bit representation of –25 obtained as 11100111.

Step 2: Add the 8-bit representations in binary notation and truncate the 1 in the 2^8th position if there is one:

$$\begin{array}{rcccccccc}
 & \boxed{0} & \boxed{0} & \boxed{1} & \boxed{0} & \boxed{0} & \boxed{1} & \boxed{1} & \boxed{1} \\
+ & & & & & & & & \\
 & \boxed{1} & \boxed{1} & \boxed{1} & \boxed{0} & \boxed{0} & \boxed{1} & \boxed{1} & \boxed{1} \\
\hline
\text{Truncate}\rightarrow \; 1 & \boxed{0} & \boxed{0} & \boxed{0} & \boxed{0} & \boxed{1} & \boxed{1} & \boxed{1} & \boxed{0}
\end{array}$$

Example 9 – *Solution*

cont'd

Step 3: Find the decimal equivalent of the result:

$$\begin{aligned} 00001110_2 &= (8+4+2)_{10} \\ &= 14_{10}. \end{aligned}$$

Since 39 – 25 = 14, this is the correct answer.

Computer Addition with Negative Integers

Case 4 (both integers are negative): This case involves adding two negative integers in the range –1 through –128 whose sum is also in this range.

To be specific, consider the sum $(-a) + (-b)$ where a, b, and $a + b$ are all in the range 1 through 128. In this case, the 8-bit representations of $-a$ and $-b$ are the 8-bit representations of $2^8 - a$ and $2^8 - b$.

So if the 8-bit representations of $-a$ and $-b$ are added, the result is

$$(2^8 - a) + (2^8 - b) = [2^8 - (a + b)] + 2^8.$$

Computer Addition with Negative Integers

We know that truncating a leading 1 in the ninth (2^8th) position of a binary number is equivalent to subtracting 2^8.

So when the leading 1 is truncated from the 8-bit representation of $(2^8 - a) + (2^8 - b)$, the result is $2^8 - (a + b)$, which is the 8-bit representation of $-(a + b) = (-a) + (-b)$.

Example 10 – *Computing (–a) + (–b) Where 1 ≤ a, b ≤ 128, and 1 ≤ a + b ≤ 128*

Use 8-bit representations to compute (–89) + (–25).

Solution:

Step 1: Change from decimal to 8-bit representations using the two's complements to represent –89 and –25.

The 8-bit representations of –89 and –25 were shown in Examples 2.5.8 and 2.5.9 to be 10100111 and 11100111, respectively.

Example 10 – *Solution*

cont’d

Step 2: Add the 8-bit representations in binary notation and truncate the 1 in the 2^8th position if there is one:

$$
\begin{array}{rcccccccccc}
 & & & \boxed{1} & \boxed{0} & \boxed{1} & \boxed{0} & \boxed{0} & \boxed{1} & \boxed{1} & \boxed{1} \\
 & + & & & & & & & & & \\
 & & & \boxed{1} & \boxed{1} & \boxed{1} & \boxed{0} & \boxed{0} & \boxed{1} & \boxed{1} & \boxed{1} \\
\hline
\text{Truncate} \rightarrow & & 1 & \boxed{1} & \boxed{0} & \boxed{0} & \boxed{0} & \boxed{1} & \boxed{1} & \boxed{1} & \boxed{0}
\end{array}
$$

Example 10 – *Solution*

cont'd

Step 3: Find the decimal equivalent of the result. Because its leading bit is 1, this number is the 8-bit representation of a negative integer.

$$10001110 \xrightarrow{\text{flip the bits}} 01110001 \xrightarrow{\text{add } 1} 01110010_2$$
$$\leftrightarrow -(64 + 32 + 16 + 2)_{10} = -114_{10}$$

Since (–89) + (–25) = –114, that is the correct answer.

Hexadecimal Notation

Hexadecimal Notation

Hexadecimal notation is even more compact than decimal notation, and it is much easier to convert back and forth between hexadecimal and binary notation than it is between binary and decimal notation.

The word *hexadecimal* comes from the Greek root *hex*-, meaning “six,” and the Latin root *deci*-, meaning “ten.” Hence *hexadecimal* refers to “sixteen,” and hexadecimal notation is also called **base 16 notation.**

Hexadecimal Notation

Hexadecimal notation is based on the fact that any integer can be uniquely expressed as a sum of numbers of the form

$$d \cdot 16^n,$$

where each *n* is a nonnegative integer and each *d* is one of the integers from 0 to 15. In order to avoid ambiguity, each hexadecimal digit must be represented by a single symbol. The integers 10 through 15 are represented by the symbols A, B, C, D, E, and F.

Hexadecimal Notation

The sixteen hexadecimal digits are shown in Table 2.5.3, together with their decimal equivalents and, for future reference, their 4-bit binary equivalents.

Decimal	Hexadecimal	4-Bit Binary Equivalent
0	0	0000
1	1	0001
2	2	0010
3	3	0011
4	4	0100
5	5	0101
6	6	0110
7	7	0111
8	8	1000
9	9	1001
10	A	1010
11	B	1011
12	C	1100
13	D	1101
14	E	1110
15	F	1111

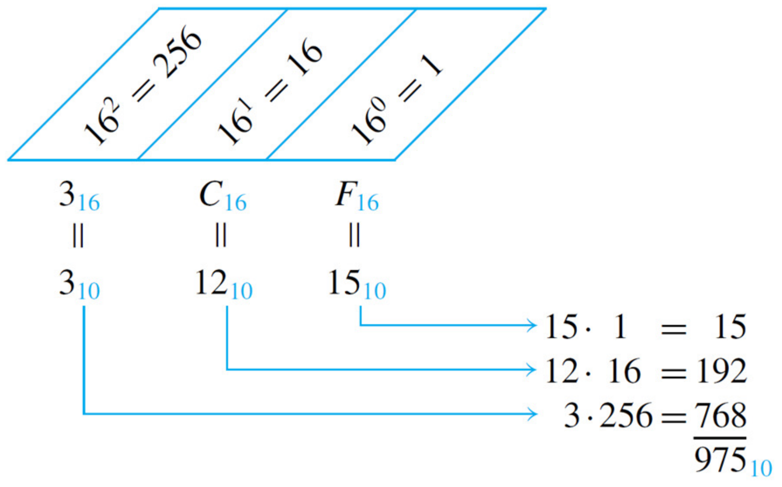
Table 2.5.3

Example 11 – *Converting from Hexadecimal to Decimal Notation*

Convert $3CF_{16}$ to decimal notation.

Solution:

Consider the following schema.

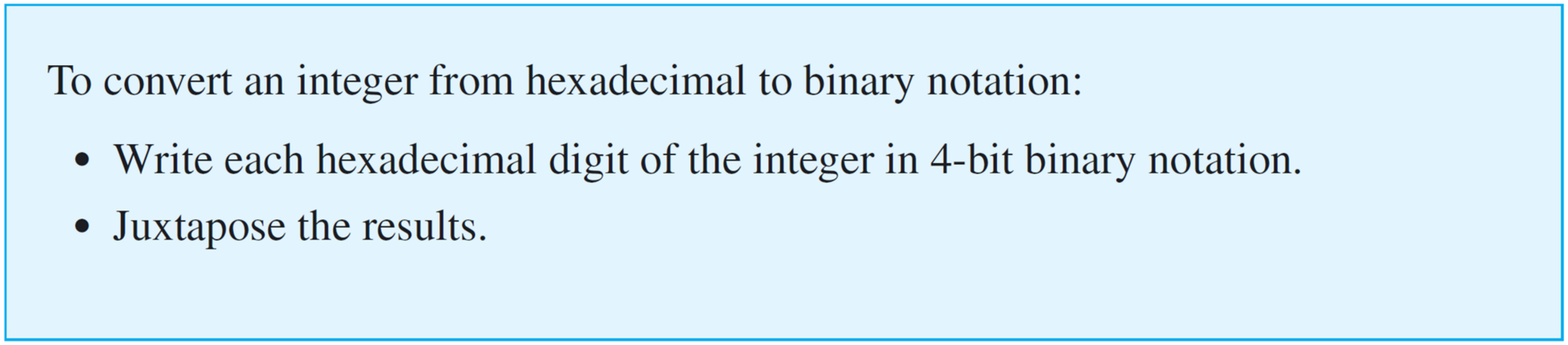


So $3CF_{16} = 975_{10}$.

Hexadecimal Notation

Now consider how to convert from hexadecimal to binary notation.

The following sequence of steps will give the required conversion from hexadecimal to binary notation.



To convert an integer from hexadecimal to binary notation:

- Write each hexadecimal digit of the integer in 4-bit binary notation.
- Juxtapose the results.

Example 12 – *Converting from Hexadecimal to Binary Notation*

Convert $B09F_{16}$ to binary notation.

Solution:

$$B_{16} = 11_{10} = 1011_2,$$

$$0_{16} = 0_{10} = 0000_2,$$

$$9_{16} = 9_{10} = 1001_2,$$

and

$$F_{16} = 15_{10} = 1111_2.$$

Example 12 – *Solution*

cont'd

Consequently,

$$\begin{tikzcd}
\mathrm{B} \arrow[d, leftrightarrow] & 0 \arrow[d, leftrightarrow] & 9 \arrow[d, leftrightarrow] & \mathrm{F} \arrow[d, leftrightarrow] \\
1011 & 0000 & 1001 & 1111
\end{tikzcd}$$

and the answer is 1011000010011111_2.

Hexadecimal Notation

To convert integers written in binary notation into hexadecimal notation, reverse the steps of the previous procedure.

To convert an integer from binary to hexadecimal notation:

- Group the digits of the binary number into sets of four, starting from the right and adding leading zeros as needed.
- Convert the binary numbers in each set of four into hexadecimal digits. Juxtapose those hexadecimal digits.

Example 13 – *Converting from Binary to Hexadecimal Notation*

Convert 100110110101001_2 to hexadecimal notation.

Solution:

First group the binary digits in sets of four, working from right to left and adding leading 0’s if necessary.

0100   1101   1010   1001.

Example 13 – *Solution*

cont'd

Convert each group of four binary digits into a hexadecimal digit.

$$\begin{array}{cccc} 0100 & 1101 & 1010 & 1001 \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ 4 & D & A & 9 \end{array}$$

Then juxtapose the hexadecimal digits.

$4DA9_{16}$