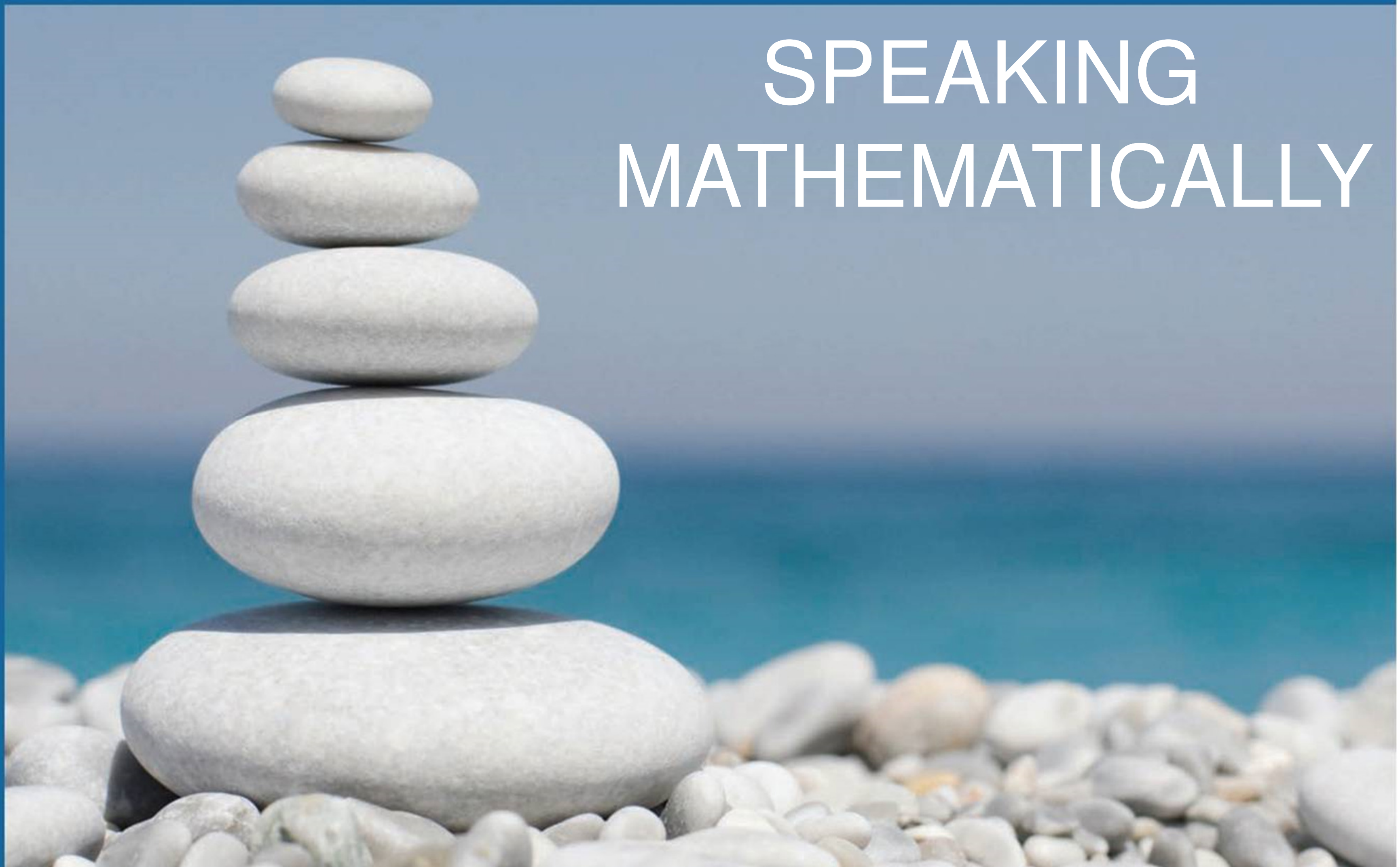
CHAPTER 1

SPEAKING MATHEMATICALLY



SECTION 1.2

The Language of Sets

The Language of Sets

The Language of Sets

Use of the word *set* as a formal mathematical term was introduced in 1879 by Georg Cantor (1845–1918). For most mathematical purposes we can think of a set intuitively, as Cantor did, simply as a collection of elements.

For instance, if C is the set of all countries that are currently in the United Nations, then the United States is an element of C, and if I is the set of all integers from 1 to 100, then the number 57 is an element of I.

The Language of Sets

• Notation

If S is a set, the notation $\boldsymbol{x} \in \boldsymbol{S}$ means that x is an element of S. The notation $\boldsymbol{x} \notin \boldsymbol{S}$ means that x is not an element of S. A set may be specified using the **set-roster notation** by writing all of its elements between braces. For example, **{1, 2, 3}** denotes the set whose elements are 1, 2, and 3. A variation of the notation is sometimes used to describe a very large set, as when we write **{1, 2, 3, . . . , 100}** to refer to the set of all integers from 1 to 100. A similar notation can also describe an infinite set, as when we write **{1, 2, 3, . . .}** to refer to the set of all positive integers. (The symbol **. . .** is called an **ellipsis** and is read "and so forth.")

The **axiom of extension** says that a set is completely determined by what its elements are—not the order in which they might be listed or the fact that some elements might be listed more than once.

Example 1 – *Using the Set-Roster Notation*

a. Let $A = \{1, 2, 3\}$, $B = \{3, 1, 2\}$, and $C = \{1, 1, 2, 3, 3, 3\}$. What are the elements of A, B, and C? How are A, B, and C related?

b. Is $\{0\} = 0$?

c. How many elements are in the set $\{1, \{1\}\}$?

d. For each nonnegative integer n, let $U_n = \{n, -n\}$. Find U_1, U_2, and U_0.

Solution:

a. A, B, and C have exactly the same three elements: 1, 2, and 3. Therefore, A, B, and C are simply different ways to represent the same set.

Example 1 – *Solution*

cont'd

b. $\{0\} \neq 0$ because $\{0\}$ is a set with one element, namely 0, whereas 0 is just the symbol that represents the number zero.

c. The set $\{1, \{1\}\}$ has two elements: 1 and the set whose only element is 1.

d. $U_1 = \{1, -1\}$, $U_2 = \{2, -2\}$, $U_0 = \{0, -0\} = \{0, 0\} = \{0\}$.

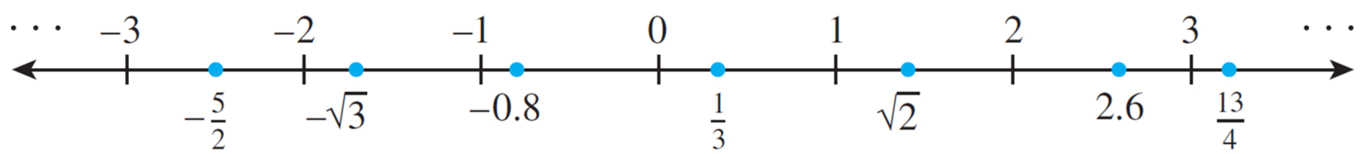
The Language of Sets

Certain sets of numbers are so frequently referred to that they are given special symbolic names. These are summarized in the following table:

Symbol	Set
R	set of all real numbers
Z	set of all integers
Q	set of all rational numbers, or quotients of integers

The Language of Sets

The set of real numbers is usually pictured as the set of all points on a line, as shown below.



The number 0 corresponds to a middle point, called the *origin*.

A unit of distance is marked off, and each point to the right of the origin corresponds to a positive real number found by computing its distance from the origin.

The Language of Sets

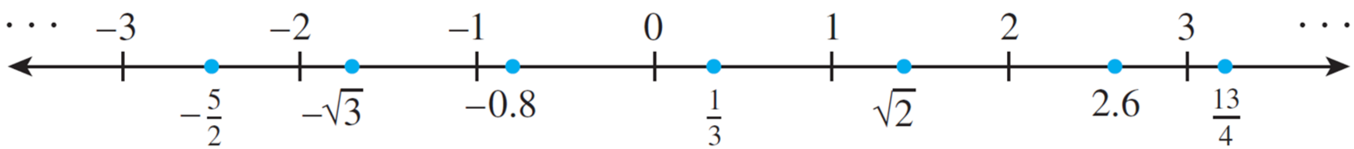
Each point to the left of the origin corresponds to a negative real number, which is denoted by computing its distance from the origin and putting a minus sign in front of the resulting number.

The set of real numbers is therefore divided into three parts: the set of positive real numbers, the set of negative real numbers, and the number 0.

Note that 0 is neither positive nor negative.

The Language of Sets

Labels are given for a few real numbers corresponding to points on the line shown below.



The real number line is called *continuous* because it is imagined to have no holes.

The set of integers corresponds to a collection of points located at fixed intervals along the real number line.

The Language of Sets

Thus every integer is a real number, and because the integers are all separated from each other, the set of integers is called *discrete*. The name *discrete mathematics* comes from the distinction between continuous and discrete mathematical objects.

Another way to specify a set uses what is called the *set-builder notation*.

• Set-Builder Notation

Let S denote a set and let $P(x)$ be a property that elements of S may or may not satisfy. We may define a new set to be **the set of all elements *x* in *S* such that *P(x)* is true**. We denote this set as follows:

$$\{x \in S \mid P(x)\}$$

the set of all ($\{$) — such that ($\mid$)

Example 2 – *Using the Set-Builder Notation*

Given that **R** denotes the set of all real numbers, **Z** the set of all integers, and $\mathbf{Z}^{+}$ the set of all positive integers, describe each of the following sets.

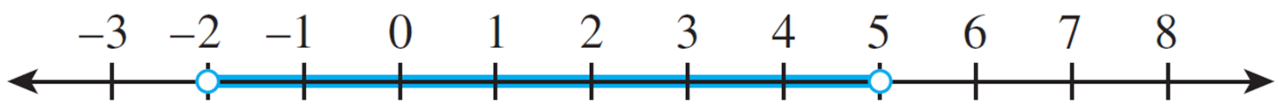
a. $\{x \in \mathbf{R} \mid -2 < x < 5\}$

b. $\{x \in \mathbf{Z} \mid -2 < x < 5\}$

c. $\{x \in \mathbf{Z}^{+} \mid -2 < x < 5\}$

Example 2 – *Solution*

a. $\{x \in \mathbf{R} \mid -2 < x < 5\}$ is the open interval of real numbers (strictly) between –2 and 5. It is pictured as follows:



b. $\{x \in \mathbf{Z} \mid -2 < x < 5\}$ is the set of all integers (strictly) between –2 and 5. It is equal to the set

{–1, 0, 1, 2, 3, 4}.

c. Since all the integers in $\mathbf{Z}^+$ are positive,

$$\{x \in \mathbf{Z}^+ \mid -2 < x < 5\} = \{1, 2, 3, 4\}.$$

Subsets

Subsets

A basic relation between sets is that of subset.

• Definition

If A and B are sets, then A is called a **subset** of B, written $\boldsymbol{A \subseteq B}$, if, and only if, every element of A is also an element of B.

Symbolically:

$A \subseteq B$ means that For all elements x, if $x \in A$ then $x \in B$.

The phrases A *is contained in* B and B *contains* A are alternative ways of saying that A is a subset of B.

Subsets

It follows from the definition of subset that for a set *A* not to be a subset of a set *B* means that there is at least one element of *A* that is not an element of *B*.

Symbolically:

$A \not\subseteq B$ means that There is at least one element x such that $x \in A$ and $x \notin B$.

• Definition

Let A and B be sets. A is a **proper subset** of B if, and only if, every element of A is in B but there is at least one element of B that is not in A.

Example 4 – *Distinction between* $\in$ *and* $\subseteq$

Which of the following are true statements?

a. $2 \in \{1, 2, 3\}$ **b.** $\{2\} \in \{1, 2, 3\}$ **c.** $2 \subseteq \{1, 2, 3\}$
d. $\{2\} \subseteq \{1, 2, 3\}$ **e.** $\{2\} \subseteq \{\{1\}, \{2\}\}$ **f.** $\{2\} \in \{\{1\}, \{2\}\}$

Solution:

Only (**a**), (**d**), and (**f**) are true.

For (**b**) to be true, the set $\{1, 2, 3\}$ would have to contain the element $\{2\}$. But the only elements of $\{1, 2, 3\}$ are 1, 2, and 3, and 2 is not equal to $\{2\}$. Hence (**b**) is false.

Example 4 – *Solution*

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For (**c**) to be true, the number 2 would have to be a set and every element in the set 2 would have to be an element of {1, 2, 3}. This is not the case, so (**c**) is false.

For (**e**) to be true, every element in the set containing only the number 2 would have to be an element of the set whose elements are {1} and {2}. But 2 is not equal to either {1} or {2}, and so (**e**) is false.

Cartesian Products

Cartesian Products

- **Notation**

Given elements a and b, the symbol (a, b) denotes the **ordered pair** consisting of a and b together with the specification that a is the first element of the pair and b is the second element. Two ordered pairs (a, b) and (c, d) are equal if, and only if, $a = c$ and $b = d$. Symbolically:

$$(a, b) = (c, d) \quad \text{means that} \quad a = c \text{ and } b = d.$$

Example 5 – *Ordered Pairs*

a. Is (1, 2) = (2, 1)?

b. Is $\left(3, \frac{5}{10}\right) = \left(\sqrt{9}, \frac{1}{2}\right)$?

c. What is the first element of (1, 1)?

Solution:

a. No. By definition of equality of ordered pairs,

(1, 2) = (2,1) if, and only if, 1 = 2 and 2 = 1.

But 1 $\neq$ 2, and so the ordered pairs are not equal.

Example 5 – *Solution*

cont'd

b. Yes. By definition of equality of ordered pairs,

$\left(3, \frac{5}{10}\right) = \left(\sqrt{9}, \frac{1}{2}\right)$ if, and only if, $3 = \sqrt{9}$ and $\frac{5}{10} = \frac{1}{2}$.

Because these equations are both true, the ordered pairs are equal.

c. In the ordered pair (1, 1), the first and the second elements are both 1.

Cartesian Products

Definition

Given sets A and B, the **Cartesian product of A and B**, denoted $\boldsymbol{A \times B}$ and read "A cross B," is the set of all ordered pairs (a, b), where a is in A and b is in B. Symbolically:

$$\boldsymbol{A \times B} = \{(a, b) \mid a \in A \text{ and } b \in B\}.$$

Example 6 – *Cartesian Products*

Let $A = \{1, 2, 3\}$ and $B = \{u, v\}$.

a. Find $A \times B$

b. Find $B \times A$

c. Find $B \times B$

d. How many elements are in $A \times B$, $B \times A$, and $B \times B$?

e. Let $\mathbf{R}$ denote the set of all real numbers. Describe $\mathbf{R} \times \mathbf{R}$.

Example 6 – *Solution*

a. $A \times B = \{(1, u), (2, u), (3, u), (1, v), (2, v), (3, v)\}$

b. $B \times A = \{(u, 1), (u, 2), (u, 3), (v, 1), (v, 2), (v, 3)\}$

c. $B \times B = \{(u, u), (u, v), (v, u), (v, v)\}$

d. $A \times B$ has six elements. Note that this is the number of elements in A times the number of elements in B.

$B \times A$ has six elements, the number of elements in B times the number of elements in A. $B \times B$ has four elements, the number of elements in B times the number of elements in B.

Example 6 – *Solution*

cont'd

e. **R** × **R** is the set of all ordered pairs (*x*, *y*) where both *x* and *y* are real numbers.

If horizontal and vertical axes are drawn on a plane and a unit length is marked off, then each ordered pair in **R** × **R** corresponds to a unique point in the plane, with the first and second elements of the pair indicating, respectively, the horizontal and vertical positions of the point.

Example 6 – *Solution*

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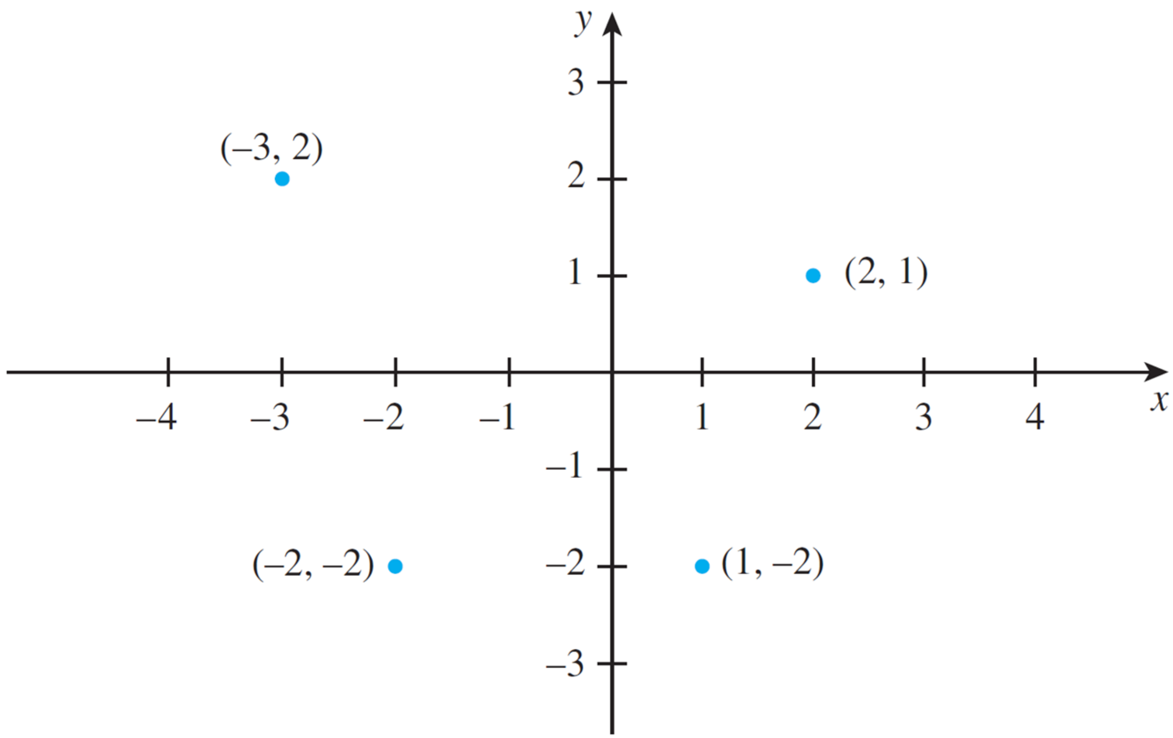
The term **Cartesian plane** is often used to refer to a plane with this coordinate system, as illustrated in Figure 1.2.1.



A Cartesian Plane

Figure 1.2.1