Solution Suppose n is an integer that is greater than or equal to 2. By Theorem 9.2.3,

$$P(n, 2) = \frac{n!}{(n-2)!} = \frac{n(n-1)\cancel{(n-2)!}}{\cancel{(n-2)!}} = n(n-1)$$

and

$$P(n, 1) = \frac{n!}{(n-1)!} = \frac{n \cdot \cancel{(n-1)!}}{\cancel{(n-1)!}} = n.$$

Hence

$$P(n, 2) + P(n, 1) = n \cdot (n-1) + n = n^2 - n + n = n^2,$$

which is what we needed to show. ■

Test Yourself

1. The multiplication rule says that if an operation can be performed in k steps and, for each i with $1 \le i \le k$, the ith step can be performed in n_i ways (regardless of how previous steps were performed), then the operation as a whole can be performed in _____.
2. A permutation of a set of elements is _____.
3. The number of permutations of a set of n elements equals _____.
4. An r-permutation of a set of n elements is _____.
5. The number of r-permutations of a set of n elements is denoted _____.
6. One formula for the number of r-permutations of a set of n elements is _____ and another formula is _____.

Exercise Set 9.2

In 1–4, use the fact that in baseball's World Series, the first team to win four games wins the series.

1. Suppose team A wins the first three games. How many ways can the series be completed? (Draw a tree.)
2. Suppose team A wins the first two games. How many ways can the series be completed? (Draw a tree.)
3. How many ways can a World Series be played if team A wins four games in a row?
4. How many ways can a World Series be played if no team wins two games in a row?
5. In a competition between players X and Y, the first player to win three games in a row or a total of four games wins. How many ways can the competition be played if X wins the first game and Y wins the second and third games? (Draw a tree.)
6. One urn contains two black balls (labeled B_1 and B_2) and one white ball. A second urn contains one black ball and two white balls (labeled W_1 and W_2). Suppose the following experiment is performed: One of the two urns is chosen at random. Next a ball is randomly chosen from the urn. Then a second ball is chosen at random from the same urn without replacing the first ball.
   a. Construct the possibility tree showing all possible outcomes of this experiment.
   b. What is the total number of outcomes of this experiment?
   c. What is the probability that two black balls are chosen?
   d. What is the probability that two balls of opposite color are chosen?
7. One urn contains one blue ball (labeled B_1) and three red balls (labeled R_1, R_2, and R_3). A second urn contains two red balls (R_4 and R_5) and two blue balls (B_2 and B_3). An experiment is performed in which one of the two urns is chosen at random and then two balls are randomly chosen from it, one after the other without replacement.
   a. Construct the possibility tree showing all possible outcomes of this experiment.
   b. What is the total number of outcomes of this experiment?
   c. What is the probability that two red balls are chosen?
8. A person buying a personal computer system is offered a choice of three models of the basic unit, two models of keyboard, and two models of printer. How many distinct systems can be purchased?
9. Suppose there are three roads from city A to city B and five roads from city B to city C.
   a. How many ways is it possible to travel from city A to city C via city B?
   b. How many different round-trip routes are there from city A to B to C to B and back to A?
   c. How many different routes are there from city A to B to C to B and back to A in which no road is traversed twice?

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10. Suppose there are three routes from North Point to Boulder Creek, two routes from Boulder Creek to Beaver Dam, two routes from Beaver Dam to Star Lake, and four routes directly from Boulder Creek to Star Lake. (Draw a sketch.)
    a. How many routes from North Point to Star Lake pass through Beaver Dam?
    b. How many routes from North Point to Star Lake bypass Beaver Dam?

11. **a.** A bit string is a finite sequence of 0's and 1's. How many bit strings have length 8?
    b. How many bit strings of length 8 begin with three 0's?
    c. How many bit strings of length 8 begin and end with a 1?

12. Hexadecimal numbers are made using the sixteen digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F. They are denoted by the subscript 16. For example, $9A2D_{16}$ and $BC54_{16}$ are hexadecimal numbers.
    a. How many hexadecimal numbers begin with one of the digits 3 through B, end with one of the digits 5 through F, and are 5 digits long?
    b. How many hexadecimal numbers begin with one of the digits 4 through D, end with one of the digits 2 through E, and are 6 digits long?

13. A coin is tossed four times. Each time the result H for heads or T for tails is recorded. An outcome of $HHTT$ means that heads were obtained on the first two tosses and tails on the second two. Assume that heads and tails are equally likely on each toss.
    a. How many distinct outcomes are possible?
    b. What is the probability that exactly two heads occur?
    c. What is the probability that exactly one head occurs?

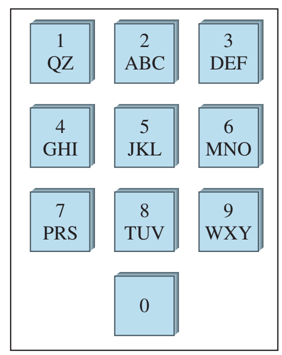
14. Suppose that in a certain state, all automobile license plates have four letters followed by three digits.
    a. How many different license plates are possible?
    b. How many license plates could begin with A and end in 0?
    c. How many license plates could begin with $TGIF$?
    d. How many license plates are possible in which all the letters and digits are distinct?
    e. How many license plates could begin with AB and have all letters and digits distinct?

15. A combination lock requires three selections of numbers, each from 1 through 30.
    a. How many different combinations are possible?
    b. Suppose the locks are constructed in such a way that no number may be used twice. How many different combinations are possible?

16. a. How many integers are there from 10 through 99?
    b. How many odd integers are there from 10 through 99?
    c. How many integers from 10 through 99 have distinct digits?
    d. How many odd integers from 10 through 99 have distinct digits?
    e. What is the probability that a randomly chosen two-digit integer has distinct digits? has distinct digits and is odd?

17. a. How many integers are there from 1000 through 9999?
    b. How many odd integers are there from 1000 through 9999?
    c. How many integers from 1000 through 9999 have distinct digits?
    d. How many odd integers from 1000 through 9999 have distinct digits?
    e. What is the probability that a randomly chosen four-digit integer has distinct digits? has distinct digits and is odd?

18. The diagram below shows the keypad for an automatic teller machine. As you can see, the same sequence of keys represents a variety of different PINs. For instance, 2133, AZDE, and BQ3F are all keyed in exactly the same way.



    a. How many different PINs are represented by the same sequence of keys as 2133?
    b. How many different PINs are represented by the same sequence of keys as 5031?
    c. At an automatic teller machine, each PIN corresponds to a four-digit numeric sequence. For instance, TWJM corresponds to 8956. How many such numeric sequences contain no repeated digit?

19. Three officers—a president, a treasurer, and a secretary—are to be chosen from among four people: Ann, Bob, Cyd, and Dan. Suppose that Bob is not qualified to be treasurer and Cyd's other commitments make it impossible for her to be secretary. How many ways can the officers be chosen? Can the multiplication rule be used to solve this problem?

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20. Modify Example 9.2.4 by supposing that a PIN must not begin with any of the letters A–M and must end with a digit. Continue to assume that no symbol may be used more than once and that the total number of PINs is to be determined.
    a. Find the error in the following "solution."

    "Constructing a PIN is a four-step process.

    Step 1: Choose the left-most symbol.

    Step 2: Choose the second symbol from the left.

    Step 3: Choose the third symbol from the left.

    Step 4: Choose the right-most symbol.

    Because none of the thirteen letters from A through M may be chosen in step 1, there are $36 - 13 = 23$ ways to perform step 1. There are 35 ways to perform step 2 and 34 ways to perform step 3 because previously used symbols may not be used. Since the symbol chosen in step 4 must be a previously unused digit, there are $10 - 3 = 7$ ways to perform step 4. Thus there are $23 \cdot 35 \cdot 34 \cdot 7 = 191{,}590$ different PINs that satisfy the given conditions."

    b. Reorder steps 1–4 in part (a) as follows:

    Step 1: Choose the right-most symbol.

    Step 2: Choose the left-most symbol.

    Step 3: Choose the second symbol from the left.

    Step 4: Choose the third symbol from the left.

    Use the multiplication rule to find the number of PINs that satisfy the given conditions.

H 21. Suppose A is a set with m elements and B is a set with n elements.
    a. How many relations are there from A to B? Explain.
    b. How many functions are there from A to B? Explain.
    c. What fraction of the relations from A to B are functions?

22. a. How many functions are there from a set with three elements to a set with four elements?
    b. How many functions are there from a set with five elements to a set with two elements?
    c. How many functions are there from a set with m elements to a set with n elements, where m and n are positive integers?

23. In Section 2.5 we showed how integers can be represented by strings of 0's and 1's inside a digital computer. In fact, through various coding schemes, strings of 0's and 1's can be used to represent all kinds of symbols. One commonly used code is the Extended Binary-Coded Decimal Interchange Code (EBCDIC) in which each symbol has an 8-bit representation. How many distinct symbols can be represented by this code?

In each of 24–28, determine how many times the innermost loop will be iterated when the algorithm segment is implemented and run. (Assume that $m, n, p, a, b, c,$ and d are all positive integers.)

24.
```
for i := 1 to 30
    for j := 1 to 15
        [Statements in body of inner loop.
        None contain branching statements that
        lead outside the loop.]
    next j
next i
```

25.
```
for j := 1 to m
    for k := 1 to n
        [Statements in body of inner loop.
        None contain branching statements that
        lead outside the loop.]
    next k
next j
```

26.
```
for i := 1 to m
    for j := 1 to n
        for k := 1 to p
            [Statements in body of inner loop.
            None contain branching statements that
            lead outside the loop.]
        next k
    next j
next i
```

27.
```
for i := 5 to 50
    for j := 10 to 20
        [Statements in body of inner loop.
        None contain branching statements that
        lead outside the loop.]
    next j
next i
```

28. Assume $a \le b$ and $c \le d$.
```
for i := a to b
    for j := c to d
        [Statements in body of inner loop.
        None contain branching statements that
        lead outside the loop.]
    next j
next i
```

H ✶ 29. Consider the numbers 1 through 99,999 in their ordinary decimal representations. How many contain exactly one of each of the digits 2, 3, 4, and 5?

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✶ 30. Let $n = p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}$ where $p_1, p_2, \ldots, p_m$ are distinct prime numbers and $k_1, k_2, \ldots, k_m$ are positive integers. How many ways can n be written as a product of two positive integers that have no common factors
   a. assuming that order matters (i.e., $8 \cdot 15$ and $15 \cdot 8$ are regarded as different)?
   b. assuming that order does not matter (i.e., $8 \cdot 15$ and $15 \cdot 8$ are regarded as the same)?

✶ 31. **a.** If p is a prime number and a is a positive integer, how many distinct positive divisors does p^a have?
   b. If p and q are distinct prime numbers and a and b are positive integers, how many distinct positive divisors does $p^a q^b$ have?
   c. If p, q, and r are distinct prime numbers and a, b, and c are positive integers, how many distinct positive divisors does $p^a q^b r^c$ have?
   d. If $p_1, p_2, \ldots, p_m$ are distinct prime numbers and $a_1, a_2, \ldots, a_m$ are positive integers, how many distinct positive divisors does $p_1^{a_1} p_2^{a_2} \cdots p_m^{a_m}$ have?
   e. What is the smallest positive integer with exactly 12 divisors?

32. **a.** How many ways can the letters of the word *ALGORITHM* be arranged in a row?
   b. How many ways can the letters of the word *ALGORITHM* be arranged in a row if *A* and *L* must remain together (in order) as a unit?
   c. How many ways can the letters of the word *ALGORITHM* be arranged in a row if the letters *GOR* must remain together (in order) as a unit?

33. Six people attend the theater together and sit in a row with exactly six seats.
   a. How many ways can they be seated together in the row?
   b. Suppose one of the six is a doctor who must sit on the aisle in case she is paged. How many ways can the people be seated together in the row with the doctor in an aisle seat?
   c. Suppose the six people consist of three married couples and each couple wants to sit together with the husband on the left. How many ways can the six be seated together in the row?

34. Five people are to be seated around a circular table. Two seatings are considered the same if one is a rotation of the other. How many different seatings are possible?

35. Write all the 2-permutations of $\{W, X, Y, Z\}$.

36. Write all the 3-permutations of $\{s, t, u, v\}$.

37. Evaluate the following quantities.
   a. $P(6, 4)$ b. $P(6, 6)$ c. $P(6, 3)$ d. $P(6, 1)$

38. **a.** How many 3-permutations are there of a set of five objects?
   b. How many 2-permutations are there of a set of eight objects?

39. **a.** How many ways can three of the letters of the word *ALGORITHM* be selected and written in a row?
   b. How many ways can six of the letters of the word *ALGORITHM* be selected and written in a row?
   c. How many ways can six of the letters of the word *ALGORITHM* be selected and written in a row if the first letter must be *A*?
   d. How many ways can six of the letters of the word *ALGORITHM* be selected and written in a row if the first two letters must be *OR*?

40. Prove that for all integers $n \geq 2$, $P(n+1, 3) = n^3 - n$.

41. Prove that for all integers $n \geq 2$,

$$P(n+1, 2) - P(n, 2) = 2P(n, 1).$$

42. Prove that for all integers $n \geq 3$,

$$P(n+1, 3) - P(n, 3) = 3P(n, 2).$$

43. Prove that for all integers $n \geq 2$, $P(n, n) = P(n, n-1)$.

44. Prove Theorem 9.2.1 by mathematical induction.

H **45.** Prove Theorem 9.2.2 by mathematical induction.

✶ 46. Prove Theorem 9.2.3 by mathematical induction.

47. A permutation on a set can be regarded as a function from the set to itself. For instance, one permutation of $\{1, 2, 3, 4\}$ is 2341. It can be identified with the function that sends each position number to the number occupying that position. Since position 1 is occupied by 2, 1 is sent to 2 or $1 \to 2$; since position 2 is occupied by 3, 2 is sent to 3 or $2 \to 3$; and so forth. The entire permutation can be written using arrows as follows:

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 2 & 3 & 4 & 1 \end{array}$$

   a. Use arrows to write each of the six permutations of $\{1, 2, 3\}$.
   b. Use arrows to write each of the permutations of $\{1, 2, 3, 4\}$ that keep 2 and 4 fixed.
   c. Which permutations of $\{1, 2, 3\}$ keep no elements fixed?
   d. Use arrows to write all permutations of $\{1, 2, 3, 4\}$ that keep no elements fixed.

Answers for Test Yourself

1. $n_1 n_2 \cdots n_k$ ways 2. an ordering of the elements of the set in a row 3. $n!$ 4. an ordered selection of r of the elements of the set 5. $P(n, r)$ 6. $n(n-1)(n-2)\cdots(n-r+1)$; $\frac{n!}{(n-r)!}$

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