Test Yourself

1. For a relation on a set to be an equivalence relation, it must be _____.
2. The notation $m \equiv n \pmod d$ is read "_____" and means that _____.
3. Given an equivalence relation R on a set A and given an element a in A, the equivalence class of a is denoted _____ and is defined to be _____.
4. If A is a set, R is an equivalence relation on A, and a and b are elements of A, then either $[a] = [b]$ or _____.
5. If A is a set and R is an equivalence relation on A, then the distinct equivalence classes of R form _____.
6. Let $A = \mathbf{Z} \times (\mathbf{Z} - \{0\})$, and define a relation R on A by specifying that for all (a, b) and (c, d) in A, $(a, b)\ R\ (c, d)$ if, and only if, $ad = bc$. Then there is exactly one equivalence class of R for each _____.

Exercise Set 8.3

1. Suppose that $S = \{a, b, c, d, e\}$ and R is a relation on S such that $a\ R\ b$, $b\ R\ c$, and $d\ R\ e$. List all of the following that must be true if R is (a) reflexive (but not symmetric or transitive), (b) symmetric (but not reflexive or transitive), (c) transitive (but not reflexive or symmetric), and (d) an equivalence relation.

$c\ R\ b \quad c\ R\ c \quad a\ R\ c \quad b\ R\ a \quad a\ R\ d \quad e\ R\ a \quad e\ R\ d \quad c\ R\ a$

2. Each of the following partitions of $\{0, 1, 2, 3, 4\}$ induces a relation R on $\{0, 1, 2, 3, 4\}$. In each case, find the ordered pairs in R.
a. $\{0, 2\}, \{1\}, \{3, 4\}$ b. $\{0\}, \{1, 3, 4\}, \{2\}$
c. $\{0\}, \{1, 2, 3, 4\}$

In each of 3–14, the relation R is an equivalence relation on the set A. Find the distinct equivalence classes of R.

3. $A = \{0, 1, 2, 3, 4\}$
$R = \{(0, 0), (0, 4), (1, 1), (1, 3), (2, 2), (3, 1), (3, 3), (4, 0), (4, 4)\}$

4. $A = \{a, b, c, d\}$
$R = \{(a, a), (b, b), (b, d), (c, c), (d, b), (d, d)\}$

5. $A = \{1, 2, 3, 4, \ldots, 20\}$. R is defined on A as follows:

$$\text{For all } x, y \in A, \quad x\ R\ y \quad \Leftrightarrow \quad 4 \mid (x - y).$$

6. $A = \{-4, -3, -2, -1, 0, 1, 2, 3, 4, 5\}$. R is defined on A as follows:

$$\text{For all } x, y \in A, \quad x\ R\ y \quad \Leftrightarrow \quad 3 \mid (x - y).$$

7. $A = \{(1, 3), (2, 4), (-4, -8), (3, 9), (1, 5), (3, 6)\}$. R is defined on A as follows: For all $(a, b), (c, d) \in A$,

$$(a, b)\ R\ (c, d) \quad \Leftrightarrow \quad ad = bc.$$

8. $X = \{a, b, c\}$ and $A = \mathscr{P}(X)$. R is defined on A as follows: For all sets U and V in $\mathscr{P}(X)$,

$$U\ R\ V \quad \Leftrightarrow \quad N(U) = N(V).$$

(That is, the number of elements in U equals the number of elements in V.)

9. $X = \{-1, 0, 1\}$ and $A = \mathscr{P}(X)$. R is defined on $\mathscr{P}(X)$ as follows: For all sets s and T in $\mathscr{P}(X)$,

$s\ R\ T \quad \Leftrightarrow \quad$ the sum of the elements in s equals the sum of the elements in T.

10. $A = \{-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5\}$. R is defined on A as follows: For all $m, n \in \mathbf{Z}$,

$$m\ R\ n \quad \Leftrightarrow \quad 3 \mid (m^2 - n^2).$$

11. $A = \{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$. R is defined on A as follows: For all $(m, n) \in A$,

$$m\ R\ n \Leftrightarrow 4 \mid (m^2 - n^2).$$

12. $A = \{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$. R is defined on A as follows: For all $(m, n) \in A$,

$$m\ R\ n \Leftrightarrow 5 \mid (m^2 - n^2).$$

13. A is the set of all strings of length 4 in a's and b's. R is defined on A as follows: For all strings s and t in A,

$s\ R\ t \quad \Leftrightarrow \quad s$ has the same first two characters as t.

14. A is the set of all strings of length 2 in 0's, 1's, and 2's. R is defined on A as follows: For all strings s and t in A,

$s\ R\ t \quad \Leftrightarrow \quad$ the sum of the characters in s equals the sum of the characters in t.

15. Determine which of the following congruence relations are true and which are false.
a. $17 \equiv 2 \pmod 5$ b. $4 \equiv -5 \pmod 7$
c. $-2 \equiv -8 \pmod 3$ d. $-6 \equiv 22 \pmod 2$

16. **a.** Let R be the relation of congruence modulo 3. Which of the following equivalence classes are equal?

$$[7], [-4], [-6], [17], [4], [27], [19]$$

b. Let R be the relation of congruence modulo 7. Which of the following equivalence classes are equal?

$$[35], [3], [-7], [12], [0], [-2], [17]$$

17. a. Prove that for all integers m and n, $m \equiv n \pmod 3$ if, and only if, $m \text{ mod } 3 = n \text{ mod } 3$.
    b. Prove that for all integers m and n and any positive integer d, $m \equiv n \pmod d$ if, and only if, $m \text{ mod } d = n \text{ mod } d$.
18. a. Give an example of two sets that are distinct but not disjoint.
    b. Find sets A_1 and A_2 and elements x, y and z such that x and y are in A_1 and y and z are in A_2 but x and z are not both in either of the sets A_1 or A_2.

In 19–31, (1) prove that the relation is an equivalence relation, and (2) describe the distinct equivalence classes of each relation.

19. A is the set of all students at your college.
    a. R is the relation defined on A as follows: For all x and y in A,

    $x\ R\ y \quad \Leftrightarrow \quad x$ has the same major (or double major) as y.

    (Assume "undeclared" is a major.)
    b. S is the relation defined on A as follows: For all $x, y \in A$,

    $$x\ S\ y \quad \Leftrightarrow \quad x \text{ is the same age as } y.$$

H 20. E is the relation defined on $\mathbf{Z}$ as follows:

$$\text{For all } m, n \in \mathbf{Z}, \quad m\ E\ n \quad \Leftrightarrow \quad 2 \mid (m - n).$$

21. F is the relation defined on $\mathbf{Z}$ as follows:

$$\text{For all } m, n \in \mathbf{Z}, \quad m\ F\ n \quad \Leftrightarrow \quad 4 \mid (m - n).$$

22. Let A be the set of all statement forms in three variables p, q, and r. $\mathbf{R}$ is the relation defined on A as follows: For all P and Q in A,

$$\text{P}\ \mathbf{R}\ \text{Q} \quad \Leftrightarrow \quad \text{P and Q have the same truth table.}$$

23. Let P be a set of parts shipped to a company from various suppliers. S is the relation defined on P as follows: For all $x, y \in P$,

$x\ S\ y \quad \Leftrightarrow \quad x$ has the same part number and is shipped from the same supplier as y.

24. Let A be the set of identifiers in a computer program. It is common for identifiers to be used for only a short part of the execution time of a program and not to be used again to execute other parts of the program. In such cases, arranging for identifiers to share memory locations makes efficient use of a computer's memory capacity. Define a relation R on A as follows: For all identifiers x and y,

$x\ R\ y \quad \Leftrightarrow \quad$ the values of x and y are stored in the same memory location during execution of the program.

25. A is the "absolute value" relation defined on $\mathbf{R}$ as follows:

$$\text{For all } x, y \in \mathbf{R}, \quad x\ A\ y \quad \Leftrightarrow \quad |x| = |y|.$$

H 26. D is the relation defined on $\mathbf{Z}$ as follows: For all $m, n \in \mathbf{Z}$,

$$m\ D\ n \quad \Leftrightarrow \quad 3 \mid (m^2 - n^2).$$

27. R is the relation defined on $\mathbf{Z}$ as follows: For all $(m, n) \in \mathbf{Z}$,

$$m\ R\ n \quad \Leftrightarrow \quad 4 \mid (m^2 - n^2).$$

28. I is the relation defined on $\mathbf{R}$ as follows:

$$\text{For all } x, y \in \mathbf{R}, \quad x\ I\ y \quad \Leftrightarrow \quad x - y \text{ is an integer.}$$

29. Define P on the set $\mathbf{R} \times \mathbf{R}$ of ordered pairs of real numbers as follows: For all $(w, x), (y, z) \in \mathbf{R} \times \mathbf{R}$,

$$(w, x)\ P\ (y, z) \quad \Leftrightarrow \quad w = y.$$

30. Define Q on the set $\mathbf{R} \times \mathbf{R}$ as follows: For all $(w, x), (y, z) \in \mathbf{R} \times \mathbf{R}$,

$$(w, x)\ Q\ (y, z) \quad \Leftrightarrow \quad x = z.$$

31. Let P be the set of all points in the Cartesian plane except the origin. R is the relation defined on P as follows: For all p_1 and p_2 in P,

$p_1\ R\ p_2 \quad \Leftrightarrow \quad p_1$ and p_2 lie on the same half-line emanating from the origin.

H 32. Let A be the set of all straight lines in the Cartesian plane. Define a relation $\|$ on A as follows:

$$\text{For all } l_1 \text{ and } l_2 \text{ in } A, \quad l_1 \parallel l_2 \quad \Leftrightarrow \quad l_1 \text{ is parallel to } l_2.$$

Then $\|$ is an equivalence relation on A. Describe the equivalence classes of this relation.

33. Let A be the set of points in the rectangle with x and y coordinates between 0 and 1. That is,

$$A = \{(x, y) \in \mathbf{R} \times \mathbf{R} \mid 0 \le x \le 1 \quad \text{and} \quad 0 \le y \le 1\}.$$

Define a relation R on A as follows: For all (x_1, y_1) and (x_2, y_2) in A,

$$\begin{aligned}
(x_1, y_1)\ R\ (x_2, y_2) \Leftrightarrow \\
&(x_1, y_1) = (x_2, y_2); \quad \text{or} \\
&x_1 = 0 \text{ and } x_2 = 1 \text{ and } y_1 = y_2; \quad \text{or} \\
&x_1 = 1 \text{ and } x_2 = 0 \text{ and } y_1 = y_2; \quad \text{or} \\
&y_1 = 0 \text{ and } y_2 = 1 \text{ and } x_1 = x_2; \quad \text{or} \\
&y_1 = 1 \text{ and } y_2 = 0 \text{ and } x_1 = x_2.
\end{aligned}$$

In other words, all points along the top edge of the rectangle are related to the points along the bottom edge directly beneath them, and all points directly opposite each other along the left and right edges are related to each other. The points in the interior of the rectangle are not related to anything other than themselves. Then R is an equivalence relation on A. Imagine gluing together all the points that are in the same equivalence class. Describe the resulting figure.

34. The documentation for the computer language Java recommends that when an "equals method" is defined for an object, it be an equivalence relation. That is, if R is defined as follows:

$$x\ R\ y \quad \Leftrightarrow \quad x.equals(y) \quad \text{for all objects in the class,}$$

then R should be an equivalence relation. Suppose that in trying to optimize some of the mathematics of a graphics application, a programmer creates an object called a point, consisting of two coordinates in the plane. The programmer defines an equals method as follows: If p and q are any points, then

$$p.equals(q) \quad \Leftrightarrow \quad \text{the distance from } p \text{ to } q \text{ is less than or equal to } c$$

where c is a small positive number that depends on the resolution of the computer display. Is the programmer's equals method an equivalence relation? Justify your answer.

35. Find an additional representative circuit for the input/output table of Example 8.3.9.

Let R be an equivalence relation on a set A. Prove each of the statements in 36–41 directly from the definitions of equivalence relation and equivalence class without using the results of Lemma 8.3.2, Lemma 8.3.3, or Theorem 8.3.4.

36. For all a in A, $a \in [a]$.

37. For all a and b in A, if $b \in [a]$ then $a\ R\ b$.

38. For all a, b and c in A, if $b\ R\ c$ and $c \in [a]$ then $b \in [a]$.

39. For all a and b in A, if $[a] = [b]$ then $a\ R\ b$.

40. For all a, b, and x in A, if $a\ R\ b$ and $x \in [a]$, then $x \in [b]$.

H **41.** For all a and b in A, if $a \in [b]$ then $[a] = [b]$.

42. Let R be the relation defined in Example 8.3.12.
   a. Prove that R is reflexive.
   b. Prove that R is symmetric.
   c. List four distinct elements in $[(1, 3)]$.
   d. List four distinct elements in $[(2, 5)]$.

✶ 43. In Example 8.3.12, define operations of addition (+) and multiplication (·) as follows: For all $(a, b), (c, d) \in A$,

$$[(a, b)] + [(c, d)] = [(ad + bc, bd)]$$
$$[(a, b)] \cdot [(c, d)] = [(ac, bd)].$$

   a. Prove that this addition is well defined. That is, show that if $[(a, b)] = [(a', b')]$ and $[(c, d)] = [(c', d')]$, then $[(ad + bc, bd)] = [(a'd' + b'c', b'd')]$.
   b. Prove that this multiplication is well defined. That is, show that if $[(a, b)] = [(a', b')]$ and $[(c, d)] = [(c', d')]$, then $[(ac, bd)] = [(a'c', b'd')]$.
   c. Show that $[(0, 1)]$ is an identity element for addition. That is, show that for any $(a, b) \in A$,
   $$[(a, b)] + [(0, 1)] = [(0, 1)] + [(a, b)] = [(a, b)].$$
   d. Find an identity element for multiplication. That is, find (i, j) in A so that for all (a, b) in A. $[(a, b)] \cdot [(i, j)] = [(i, j)] \cdot [(a, b)] = [(a, b)]$.
   e. For any $(a, b) \in A$, show that $[(-a, b)]$ is an inverse for $[(a, b)]$ for addition. That is, show that $[(-a, b)] + [(a, b)] = [(a, b)] + [(-a, b)] = [(0, 1)]$.
   f. Given any $(a, b) \in A$ with $a \neq 0$, find an inverse for $[(a, b)]$ for multiplication. That is, find (c, d) in A so that $[(a, b)] \cdot [(c, d)] = [(c, d)] - [(a, b)] = [(i, j)]$, where $[(i, j)]$ is the identity element you found in part (d).

44. Let $A = \mathbf{Z}^+ \times \mathbf{Z}^+$. Define a relation R on A as follows: For all (a, b) and (c, d) in A,

$$(a, b)\ R\ (c, d) \quad \Leftrightarrow \quad a + d = c + b.$$

   a. Prove that R is reflexive.
   b. Prove that R is symmetric.
   H **c.** Prove that R is transitive.
   d. List five elements in $[(1, 1)]$.
   e. List five elements in $[(3, 1)]$.
   f. List five elements in $[(1, 2)]$.
   g. Describe the distinct equivalence classes of R.

45. The following argument claims to prove that the requirement that an equivalence relation be reflexive is redundant. In other words, it claims to show that if a relation is symmetric and transitive, then it is reflexive. Find the mistake in the argument.

"**Proof:** Let R be a relation on a set A and suppose R is symmetric and transitive. For any two elements x and y in A, if $x\ R\ y$ then $y\ R\ x$ since R is symmetric. But then it follows by transitivity that $x\ R\ x$. Hence R is reflexive."

46. Let R be a relation on a set A and suppose R is symmetric and transitive. Prove the following: If for every x in A there is a y in A such that $x\ R\ y$, then R is an equivalence relation.

47. Refer to the quote at the beginning of this section to answer the following questions.
   a. What is the name of the Knight's song called?
   b. What is the name of the Knight's song?
   c. What is the Knight's song called?
   d. What *is* the Knight's song?
   e. What is your (full, legal) name?
   f. What are you called?
   g. What *are* you? (Do not answer this on paper; just think about it.)

Answers for Test Yourself

1. reflexive, symmetric, and transitive 2. m is congruent to n modulo d; d divides $m - n$ 3. $[a]$; the set of all x in A such that $x R a$ 4. $[a] \cap [b] = \emptyset$ 5. a partition of A 6. rational number