8. To show that a relation R on a set A is not symmetric, you ______.

9. To show that a relation R on a set A is not transitive, you ______.

10. Given a relation R on a set A, the transitive closure of R is the relation R^t on A that satisfies the following three properties: ______, ______, and ______.

Exercise Set 8.2

In 1–8 a number of relations are defined on the set $A = \{0, 1, 2, 3\}$. For each relation:
a. Draw the directed graph.
b. Determine whether the relation is reflexive.
c. Determine whether the relation is symmetric.
d. Determine whether the relation is transitive.
Give a counterexample in each case in which the relation does not satisfy one of the properties.

1. $R_1 = \{(0, 0), (0, 1), (0, 3), (1, 1), (1, 0), (2, 3), (3, 3)\}$
2. $R_2 = \{(0, 0), (0, 1), (1, 1), (1, 2), (2, 2), (2, 3)\}$
3. $R_3 = \{(2, 3), (3, 2)\}$
4. $R_4 = \{(1, 2), (2, 1), (1, 3), (3, 1)\}$
5. $R_5 = \{(0, 0), (0, 1), (0, 2), (1, 2)\}$
6. $R_6 = \{(0, 1), (0, 2)\}$
7. $R_7 = \{(0, 3), (2, 3)\}$
8. $R_8 = \{(0, 0), (1, 1)\}$

In 9–33 determine whether the given relation is reflexive, symmetric, transitive, or none of these. Justify your answers.

9. R is the "greater than or equal to" relation on the set of real numbers: For all $x, y \in \mathbf{R}$, $x \; R \; y \Leftrightarrow x \geq y$.

10. C is the circle relation on the set of real numbers: For all $x, y \in \mathbf{R}$, $x \; C \; y \Leftrightarrow x^2 + y^2 = 1$.

11. D is the relation defined on $\mathbf{R}$ as follows: For all $x, y \in \mathbf{R}$, $x \; D \; y \Leftrightarrow xy \geq 0$.

12. E is the congruence modulo 2 relation on $\mathbf{Z}$: For all $m, n \in \mathbf{Z}$, $m \; E \; n \Leftrightarrow 2 \mid (m - n)$.

13. F is the congruence modulo 5 relation on $\mathbf{Z}$: For all $m, n \in \mathbf{Z}$, $m \; F \; n \Leftrightarrow 5 \mid (m - n)$.

14. O is the relation defined on $\mathbf{Z}$ as follows: For all $m, n \in \mathbf{Z}$, $m \; O \; n \Leftrightarrow m - n$ is odd.

15. D is the "divides" relation on $\mathbf{Z}^+$: For all positive integers m and n, $m \; D \; n \Leftrightarrow m \mid n$.

16. A is the "absolute value" relation on $\mathbf{R}$: For all real numbers x and y, $x \; A \; y \Leftrightarrow |x| = |y|$.

17. Recall that a prime number is an integer that is greater than 1 and has no positive integer divisors other than 1 and itself. (In particular, 1 is not prime.) A relation P is defined on $\mathbf{Z}$ as follows: For all $m, n \in \mathbf{Z}$, $m \; P \; n \Leftrightarrow \exists$ a prime number p such that $p \mid m$ and $p \mid n$.

H 18. Define a relation Q on $\mathbf{R}$ as follows: For all real numbers x and y, $x \; Q \; y \Leftrightarrow x - y$ is rational.

19. Define a relation I on $\mathbf{R}$ as follows: For all real numbers x and y, $x \; I \; y \Leftrightarrow x - y$ is irrational.

20. Let $X = \{a, b, c\}$ and $\mathscr{P}(X)$ be the power set of X (the set of all subsets of X). A relation **E** is defined on $\mathscr{P}(X)$ as follows: For all $A, B \in \mathscr{P}(X)$, $A \; \mathbf{E} \; B \Leftrightarrow$ the number of elements in A equals the number of elements in B.

21. Let $X = \{a, b, c\}$ and $\mathscr{P}(X)$ be the power set of X. A relation **L** is defined on $\mathscr{P}(X)$ as follows: For all $A, B \in \mathscr{P}(X)$, $A \; \mathbf{L} \; B \Leftrightarrow$ the number of elements in A is less than the number of elements in B.

22. Let $X = \{a, b, c\}$ and $\mathscr{P}(X)$ be the power set of X. A relation **N** is defined on $\mathscr{P}(X)$ as follows: For all $A, B \in \mathscr{P}(X)$, $A \; \mathbf{N} \; B \Leftrightarrow$ the number of elements in A is not equal to the number of elements in B.

23. Let X be a nonempty set and $\mathscr{P}(X)$ the power set of X. Define the "subset" relation **S** on $\mathscr{P}(X)$ as follows: For all $A, B \in \mathscr{P}(X)$, $A \; \mathbf{S} \; B \Leftrightarrow A \subseteq B$.

24. Let X be a nonempty set and $\mathscr{P}(X)$ the power set of X. Define the "not equal to" relation **U** on $\mathscr{P}(X)$ as follows: For all $A, B \in \mathscr{P}(X)$, $A \; \mathbf{U} \; B \Leftrightarrow A \neq B$.

25. Let A be the set of all strings of a's and b's of length 4. Define a relation R on A as follows: For all $s, t \in A$, $s \; R \; t \Leftrightarrow s$ has the same first two characters as t.

26. Let A be the set of all strings of 0's, 1's and 2's of length 4. Define a relation R on A as follows: For all $s, t \in A$, $s \; R \; t \Leftrightarrow$ the sum of the characters in s equals the sum of the characters in t.

27. Let A be the set of all English statements. A relation **I** is defined on A as follows: For all $p, q \in A$,

$$p \; \mathbf{I} \; q \Leftrightarrow p \to q \text{ is true.}$$

28. Let $A = \mathbf{R} \times \mathbf{R}$. A relation **F** is defined on A as follows: For all (x_1, y_1) and (x_2, y_2) in A,

$$(x_1, y_1) \; \mathbf{F} \; (x_2, y_2) \Leftrightarrow x_1 = x_2.$$

29. Let $A = \mathbf{R} \times \mathbf{R}$. A relation **S** is defined on A as follows: For all (x_1, y_1) and (x_2, y_2) in A,

$$(x_1, y_1) \; \mathbf{S} \; (x_2, y_2) \Leftrightarrow y_1 = y_2.$$

30. Let A be the "punctured plane"; that is, A is the set of all points in the Cartesian plane except the origin $(0, 0)$. A relation R is defined on A as follows: For all p_1 and p_2 in A, $p_1 \; R \; p_2 \Leftrightarrow p_1$ and p_2 lie on the same half line emanating from the origin.

31. Let A be the set of people living in the world today. A relation R is defined on A as follows: For all $p, q \in A$,
$p \; R \; q \Leftrightarrow p$ lives within 100 miles of q.

32. Let A be the set of all lines in the plane. A relation R is defined on A as follows: For all l_1 and l_2 in A, $l_1 \; R \; l_2 \Leftrightarrow l_1$ is parallel to l_2. (Assume that a line is parallel to itself.)

33. Let A be the set of all lines in the plane. A relation R is defined on A as follows: For all l_1 and l_2 in A,
$l_1 \; R \; l_2 \Leftrightarrow l_1$ is perpendicular to l_2.

In 34–36, assume that R is a relation on a set A. Prove or disprove each statement.

34. If R is reflexive, then R^{-1} is reflexive.

35. If R is symmetric, then R^{-1} is symmetric.

36. If R is transitive, then R^{-1} is transitive.

In 37–42, assume that R and S are relations on a set A. Prove or disprove each statement.

37. If R and S are reflexive, is $R \cap S$ reflexive? Why?

***H*38.** If R and S are symmetric, is $R \cap S$ symmetric? Why?

39. If R and S are transitive, is $R \cap S$ transitive? Why?

40. If R and S are reflexive, is $R \cup S$ reflexive? Why?

41. If R and S are symmetric, is $R \cup S$ symmetric? Why?

42. If R and S are transitive, is $R \cup S$ transitive? Why?

In 43–50 the following definitions are used: A relation on a set A is defined to be

irreflexive if, and only if, for all $x \in A$, $x \not R \, x$;

asymmetric if, and only if, for all $x, y \in A$, if $x \; R \; y$ then $y \not R \, x$;

intransitive if, and only if, for all $x, y, z \in A$, if $x \; R \; y$ and $y \; R \; z$ then $x \not R \, z$.

For each of the relations in the referenced exercise, determine whether the relation is irreflexive, asymmetric, intransitive, or none of these.

43. Exercise 1

44. Exercise 2

45. Exercise 3

46. Exercise 4

47. Exercise 5

48. Exercise 6

49. Exercise 7

50. Exercise 8

In 51–53. R. S. and T are relations defined on $A = \{0, 1, 2, 3\}$.

51. Let $R = \{(0, 1), (0, 2), (1, 1), (1, 3), (2, 2), (3, 0)\}$. Find R^t, the transitive closure of R.

52. Let $S = \{(0, 0), (0, 3), (1, 0), (1, 2), (2, 0), (3, 2)\}$. Find S^t, the transitive closure of S.

53. Let $T = \{(0, 2), (1, 0), (2, 3), (3, 1)\}$. Find T^t, the transitive closure of T.

54. Write a computer algorithm to test whether a relation R defined on a finite set A is reflexive, where $A = \{a[1], a[2], \ldots, a[n]\}$.

55. Write a computer algorithm to test whether a relation R defined on a finite set A is symmetric, where $A = \{a[1], a[2], \ldots, a[n]\}$.

56. Write a computer algorithm to test whether a relation R defined on a finite set A is transitive, where $A = \{a[1], a[2], \ldots, a[n]\}$.

Answers for Test Yourself

1. for all x in A, $x \; R \; x$ 2. for all x and y in A, if $x \; R \; y$ then $y \; R \; x$ 3. for all x, y, and z in A, if $x \; R \; y$ and $y \; R \; z$ then $x \; R \; z$ 4. x is any element of A; $x \; R \; x$ 5. x and y are any elements of A such that $x \; R \; y$; $y \; R \; x$ 6. x, y, and z are any elements of A such that $x \; R \; y$ and $y \; R \; z$; $x \; R \; z$ 7. show that there is an element x in A such that $x \not R \, x$ 8. show that there are elements x and y in A such that $x \; R \; y$ but $y \not R \, x$ 9. show that there are elements x, y, and z in A such that $x \; R \; y$ and $y \; R \; z$ but $x \not R \, z$ 10. R^t is transitive; $R \subseteq R^t$; if S is any other transitive relation that contains R, then $R^t \subseteq S$

8.3 Equivalence Relations

"You are sad" the Knight said in an anxious tone: "let me sing you a song to comfort you."

"Is it very long?" Alice asked, for she had heard a good deal of poetry that day.

"It's long," said the Knight, "but it's very, very beautiful. Everybody that hears me sing it—either it brings the tears into the eyes, or else—"

"Or else what?" said Alice, for the Knight had made a sudden pause.

"Or else it doesn't, you know. The name of the song is called 'Haddocks' Eyes.' "